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Solvability of singular integro-differential equations via Riemann–Hilbert problem ☆

Pingrun Li [a], Guangbin Ren [b,*]

[a] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China*
[b] *School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China*

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Abstract

The article is to study singular integro-differential equations involving convolutional operators and Cauchy integral operators via Riemann–Hilbert problem. To do this, we adopt a new approach through Fourier transform on L^2 subspace which is Hölder-continuous with a certain decay at infinity. The Fourier transform converts the equations into a Riemann–Hilbert problem with Hölder-continuous coefficients and with nodal points, which allows us to construct the general solutions.

MSC: 45E10; 45E05; 47G20; 30E25

Keywords: Singular integro-different equations; Riemann–Hilbert problem; Cauchy kernels; Convolution kernel

1. Introduction

The theory of singular integro-differential equations has been widely studied. We refer to [13] for the pioneer work of the topic and [20,16] for the fundamental theory. In this area there arises many approaches, such as the numerical approach [10], Lie group approach [6], and the Fredholm operator approach [22]. Recently, much attention is focused on equations of convolution kernels with or without continuous coefficients; see ([1,21,3,14,15,19,27]).

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* Corresponding author.
E-mail addresses: lipingrun@163.com (P. Li), rengb@ustc.edu.cn (G. Ren).

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The purpose of this article is to study the singular integro-different equations with convolution kernels of the form:

$$\sum_{j=0}^{n}\left\{a_j\omega^{(j)}(t)+\frac{b_j}{\pi i}\int_{\mathbb{R}}\frac{\omega^{(j)}(s)}{s-t}ds+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^+}k_{j,1}(t-s)\omega^{(j)}(s)ds\right.$$
$$+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^+}h_{j,1}(t-s)\mathcal{T}\omega^{(j)}(s)ds+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^-}k_{j,2}(t-s)\omega^{(j)}(s)ds$$
$$\left.+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^-}h_{j,2}(t-s)\mathcal{T}\omega^{(j)}(s)ds\right\}=d(t), \qquad \forall\, t\in\mathbb{R}.$$

Here ω is the unknown function and $\omega^{(j)}$ is its derivative of order j, $\mathcal{T}$ is the Cauchy integral transform.

Such kind of equations appear frequently in practical problems, such as fluid dynamics, shell theory, underwater acoustics, elasticity theory, and quantum mechanics ([4,2,12]).

We shall adopt a new approach in this article. We shall use Fourier transforms to the subspace of $L^2(\mathbb{R})$ in which functions are Hölder-continuous with a certain decay at infinity. This allows us to transform the problems of solving equations into boundary value problems for analytic function with discontinuous coefficients. By means of the classical Riemann boundary value problem, and of the principle of analytic continuation, we can prove the existence of the solution under certain conditions.

This work is organized as follows. In section 2 we introduce the function classes {0} and {{0}} and study the properties of the Fourier transforms and Cauchy transforms acting on them. In section 3 we adopt the Fourier transform approach to convert the equation into a Riemann–Hilbert problem with Hölder-continuous coefficients and with nodal points. In section 4 we establish the Plemelj formula for Fourier transform. This plays a crucial role in converting the equation into the boundary value problems. Finally, in section 5 we use the Cayley transform to identify the real line as a circle so that the Plemelj formula can be applied to study the Riemann–Hilbert problem. This allows us to show that the equation can be solved under certain conditions.

2. Fourier transform and Cauchy transform in Hölder spaces

Some notation and useful lemmas are given in this section. See [7,8,22,26].

Definition 2.1. Let $\widehat{H}$ stand for the Hölder space in the extended real line $\overline{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$, in the sense that there exist positive constants A, M, and $\mu\in(0,1]$ such that

$$|f(x_1)-f(x_2)|\le A\,|x_1-x_2|^{\mu}, \qquad \forall\, x_1,x_2\in[-M,-M];$$
$$|f(x_1)-f(x_2)|\le A\left|\frac{1}{x_1}-\frac{1}{x_2}\right|^{\mu}, \qquad \forall\, x_1,x_2\in\mathbb{R}\backslash[-M,-M].$$

Definition 2.2. Let H stand for the Hölder space in the circle

$$\Gamma=\{z\in\mathbb{C}:|z+i/2|=1/2\}.$$

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