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Asymptotics of spectral quantities of Zakharov–Shabat operators

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Abstract

In this paper we provide asymptotic expansions of various spectral quantities of Zakharov–Shabat operators on the circle with new error estimates. These estimates hold uniformly on bounded subsets of potentials in Sobolev spaces.
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1. Introduction

In this paper we prove asymptotic estimates of various spectral quantities of Zakharov–Shabat (ZS) operators

$$L(\varphi) = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}$$

in one space dimension. These operators were introduced by Zakharov and Shabat [11] in connection with the Lax pair formulation of the focusing and defocusing non-linear Schrödinger equations (NLS). As a result, in appropriate function spaces of potentials on the line and on the circle, the spectrum of $L(\varphi)$ remains invariant with respect to the solutions of these equations.

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In this paper we concentrate our attention on the periodic case only. Spectral quantities of the ZS-operator are a key ingredient for constructing a non-linear version of the Fourier transform with the property that it linearizes these nonlinear equations and yields coordinates allowing to study perturbations (see [3] for a detailed construction of such a transformation in the case of the defocusing NLS equation). To state the results of this paper we first need to introduce some more notation. We assume that $\varphi = (\varphi_1, \varphi_2)$ is in $H_c^N = H^N \times H^N$, $N \in \mathbb{Z}_{\geq 0}$, where H^N denotes the Sobolev space of 1-periodic complex-valued functions supplied with the standard Sobolev norm $\|u\|_{H^N} := \big(\sum_{j=0}^N \|\partial_x^j u\|_{L^2}^2\big)^{1/2}$, $\|u\|_{L^2} := \big(\int_0^1 |u(x)|^2\,dx\big)^{1/2}$. We consider the operator $L(\varphi)$ with periodic, anti-periodic, and Dirichlet boundary conditions on $[0, 1]$. In order to treat the cases of periodic and anti-periodic boundary conditions simultaneously consider for a given potential $\varphi \in H_c^0 \equiv L_c^2$ the operator $L(\varphi)$ with periodic boundary conditions on the interval $[0, 2]$. This operator has a compact resolvent and a discrete spectrum so that each eigenvalue admits an eigenfunction which is either periodic or anti-periodic of period 1. Following the well established tradition we call this spectrum *periodic*. Note that unless $\varphi_2 = \overline{\varphi_1}$ the operator $L(\varphi)$ is *not* formally selfadjoint with respect to the L^2-inner product on $[0, 2]$,

$$\langle F, G\rangle = \frac{1}{2}\int_0^2 \big(F_1\overline{G_1} + F_2\overline{G_2}\big)\,dx,$$

where $F = (F_1, F_2)$ and $G = (G_1, G_2)$ are complex-valued L^2-functions on $[0, 2]$. In addition, we will also consider $L(\varphi)$ with *Dirichlet* boundary conditions on $[0, 1]$ whose domain consists of all functions $F = (F_1, F_2)$ in $H^1([0, 1], \mathbb{C}) \times H^1([0, 1], \mathbb{C})$ such that

$$F_1(0) = F_2(0),\ F_1(1) = F_2(1)\,.$$

We remark that when $L(\varphi)$ is written in the form of an AKNS operator, the boundary conditions above correspond to the standard Dirichlet boundary conditions – see [3]. The corresponding spectra is referred to as the *Dirichlet spectrum* of $L(\varphi)$. It is well known that the periodic eigenvalues can be listed (with their algebraic multiplicities) as a sequence of complex numbers

$$\cdots \preceq \lambda_n^- \preceq \lambda_n^+ \preceq \lambda_{n+1}^- \preceq \lambda_{n+1}^+ \preceq \cdots$$

in lexicographic order $\preceq$ in such a way that

$$\lambda_n^-, \lambda_n^+ = n\pi + \ell_n^2 \quad \text{as} \quad |n| \to \infty \tag{1}$$

– see e.g. [3], Proposition 5.3. The notation $\lambda_n^+ = n\pi + \ell_n^2$ means that $(\lambda_n^+ - n\pi)_{n\in\mathbb{Z}}$ is an ℓ^2-sequence. Throughout this paper two complex numbers a and b are called *lexicographically* ordered $a \preceq b$, if $[\mathrm{Re}(a) < \mathrm{Re}(b)]$ or $[\mathrm{Re}(a) = \mathrm{Re}(b)$ and $\mathrm{Im}(a) \leq \mathrm{Im}(b)]$. Note that according to (1), for $|n| > N$ with $N > 0$ sufficiently large, λ_n^- and λ_n^+ are the two eigenvalues of $L(\varphi)$ close to $n\pi$. For the finitely many remaining indices n with $|n| \leq N$, no such characterization of the pair λ_n^-, λ_n^+ is possible. Similarly, the Dirichlet eigenvalues can be listed (with their algebraic multiplicities) as

$$\cdots \preceq \mu_n \preceq \mu_{n+1} \preceq \cdots$$

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