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Monotone dynamical systems with dense periodic points

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Abstract

In this paper we prove a recent conjecture by M. Hirsch, which says that if (f, Ω) is a discrete time monotone dynamical system, with $f : \Omega \to \Omega$ a homeomorphism on an open connected subset of a finite dimensional vector space, and the periodic points of f are dense in Ω, then f is periodic.

MSC: primary 37C25; secondary 47H07

Keywords: Chaos; Dense periodic points; Monotone dynamical systems

1. Introduction

The theory of monotone dynamical systems concerns the behaviour of dynamical systems on subsets of real vector spaces that preserve a partial ordering induced by a cone in the vector space. Pioneering studies by M. Hirsch [2–4] and numerous subsequent works, see [1,4,6,7, 9,11] and the references therein, showed that under suitable additional conditions the generic behaviour of such dynamical systems cannot be very complex. Common additional conditions include smoothness conditions on the system and various strong forms of monotonicity.

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In this paper we will consider discrete time dynamical systems (f, Ω), where Ω is an open connected subset of a finite dimensional real vector space V and $f: \Omega \to \Omega$ is a homeomorphism which is monotone with respect to a solid closed cone $C \subseteq V$: so, $f(x) \leq_C f(y)$ if $x \leq_C y$ and $x, y \in \Omega$. For such discrete dynamical systems the complexity of the generic behaviour is not well understood. Recently, however, M. Hirsch [5] showed that if the cone C is polyhedral, then the system cannot display chaotic behaviour in the following sense. He showed that if such a monotone dynamical system (f, Ω) has a dense set of periodic points in Ω, then f is periodic, that is to say, there exists an integer $p \geq 1$ such that $f^p(x) = x$ for all $x \in \Omega$. Furthermore, he conjectured that this result holds for general solid closed cones in finite dimensional vector spaces. The main purpose of this paper is to confirm this conjecture.

As in [5] we will also use the following theorem, which is a direct consequence of Montgomery [8].

Theorem 1.1 *(Montgomery). If $f: \Omega \to \Omega$ is a homeomorphism of an open connected subset Ω of a finite dimensional real vector space V and each $x \in \Omega$ is a periodic point of f, then f is periodic.*

2. Preliminaries

We begin by recalling some basic terminology. Throughout the paper V will be a finite dimensional real vector space equipped with the usual norm topology. A *cone* $C \subseteq V$ is a convex set such that $\lambda C \subseteq C$ for all $\lambda \geq 0$ and $C \cap -C = \{0\}$. The cone is said to be *solid* if the interior of C, denoted C°, is non-empty. We will only be working with solid closed cones. Given a solid closed cone C, the *dual cone* of C is given by $C^* = \{\varphi \in V^*: \varphi(x) \geq 0 \text{ for all } x \in C\}$, which is also solid and closed. Let us fix $u \in C^\circ$. Then the *state space* of C is defined by

$$S_C = \{\varphi \in C^*: \varphi(u) = 1\},$$

which is a compact convex set that spans V^*. We write ∂S_C to denote the boundary of S_C with respect to its affine span.

The cone C induces a partial ordering $\leq_C$ on V by $x \leq_C y$ if $y - x \in C$. We shall write $x <_C y$ if $x \leq_C y$ and $x \neq y$, and write $x \ll_C y$ if $y - x \in C^\circ$. Given $x, z \in V$ we define the *order-interval* $[x, z]_C := \{y \in V: x \leq_C y \leq_C z\}$. Note that if $x \ll_C z$, then $[x, z]_C$ is a compact, convex set with non-empty interior, denoted $[x, z]_C^\circ$, and $[x, z]_C^\circ = \{y \in V: x \ll_C y \ll_C z\}$. A map $f: \Omega \to \Omega$, where $\Omega \subseteq V$, is said to be *monotone* (with respect to a cone C), if $x \leq_C y$ and $x, y \in \Omega$ implies $f(x) \leq_C f(y)$.

We also use the following basic dynamical systems terminology. We denote a discrete time dynamical system by a pair (f, Ω), so $f: \Omega \to \Omega$. We say that $x \in \Omega$ is a *periodic point* in (f, Ω) if there exists an integer $p \geq 1$ such that $f^p(x) = x$. The least such $p \geq 1$ is called the *period* of x. A periodic point with period 1 is called a *fixed point*. The set of all periodic points of (f, Ω) is denoted $\mathrm{Per}(f)$, and the set of all fixed points of f is denoted $\mathrm{Fix}(f)$. A periodic point $x \in \Omega$ of f with period p is said to be *stable* if there exists a neighbourhood $U \subseteq \Omega$ of x such that $f^p(U) \subseteq U$.

Lemma 2.1. *If (f, Ω) be a monotone dynamical system, where $f: \Omega \to \Omega$ is a homeomorphism and $\mathrm{Per}(f)$ is dense in Ω, then*

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