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Well-posedness of parabolic equations in the non-reflexive and anisotropic Musielak–Orlicz spaces in the class of renormalized solutions [☆]

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Abstract

We prove existence and uniqueness of renormalized solutions to general nonlinear parabolic equation in Musielak–Orlicz space avoiding growth restrictions. Namely, we consider

$$\partial_t u - \operatorname{div} A(x, \nabla u) = f \in L^1(\Omega_T),$$

on a Lipschitz bounded domain in $\mathbb{R}^N$. The growth of the weakly monotone vector field A is controlled by a generalized nonhomogeneous and anisotropic N-function M. The approach does not require any particular type of growth condition of M or its conjugate M^* (neither Δ_2, nor ∇_2). The condition we impose on M is continuity of log-Hölder-type, which results in good approximation properties of the space. However, the requirement of regularity can be skipped in the case of reflexive spaces. The proof of the main results uses truncation ideas, the Young measures methods and monotonicity arguments. Uniqueness results from the comparison principle.

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1. Introduction

Our aim is to find a way of proving the existence and uniqueness of renormalized solutions to a strongly nonlinear parabolic equation with L^1-data under minimal restrictions on the growth of the leading part of the operator. We investigate operators A, which are monotone, but not necessarily strictly monotone. The modular function M, which controls the growth of the operator, is not assumed to be isotropic, i.e. $M = M(x,\xi)$ not only $M = M(x,|\xi|)$. In turn, we can expect different behavior of $M(x,\cdot)$ in various directions. We **do not** require $M \in \Delta_2$, nor $M^* \in \Delta_2$, nor any particular growth of M, such as $M(x,\xi) \geq c|\xi|^{1+\nu}$ for $\xi > \xi_0$. In general, if the modular function has a growth of type far from being polynomial (e.g. exponential), it entails analytical difficulties and significantly restricts good properties of the space, such as separability or reflexivity, as well as admissible classical tools. In order to relax the conditions on the growth we require the log-Hölder-type regularity of the modular function (cf. condition (M)), which can be skipped in reflexive spaces.

We study the problem

$$\begin{cases} \partial_t u - \operatorname{div} A(x,\nabla u) = f(t,x) & \text{in } \Omega_T, \\ u(t,x) = 0 & \text{on } \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $[0,T]$ is a finite interval, Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $\Omega_T = (0,T)\times\Omega$, $N>1$, $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$, within two classes of functions:

$$\begin{aligned} V_T^M(\Omega) &= \{u \in L^1(0,T;W_0^{1,1}(\Omega)) : \ \nabla u \in L_M(\Omega_T;\mathbb{R}^N)\}, \\ V_T^{M,\infty}(\Omega) &= V_T^M(\Omega) \cap L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

The space L_M (Definition 2.1) is equipped with the modular function M being an N-function (Definition A.1) controlling the growth of A.

We consider A belonging to an Orlicz class with respect to the second variable. Namely, we assume that function $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions.

(A1) A is a Carathéodory's function.

(A2) There exists an N-function $M : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and a constant $c_A \in (0,1]$ such that for all $\xi \in \mathbb{R}^N$ we have

$$A(x,\xi)\xi \geq c_A\left(M(x,\xi) + M^*(x,A(x,\xi))\right),$$

where M^* is conjugate to M (see Definition A.2).

(A3) For all $\xi,\eta \in \mathbb{R}^N$ and $x \in \Omega$ we have

$$(A(x,\xi) - A(x,\eta))\cdot(\xi-\eta) \geq 0.$$

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