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Cyclicity of polynomial nondegenerate centers on center manifolds ☆

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Abstract

We consider polynomial families of ordinary differential equations on $\mathbb{R}^3$, parametrized by the admissible coefficients, for which the origin is an isolated singularity at which the linear part of the system has one non-zero real and two purely imaginary eigenvalues. We derive theorems that bound the maximum number of limit cycles within the center manifold that can bifurcate, under arbitrarily small perturbation of the coefficients, from any center at the origin. The bounds are global in that they apply to the system corresponding to any point on any irreducible component of the center variety in the space of parameters. We also derive theorems for such bounds when attention is confined to a single irreducible component of the center variety.

MSC: 37G15; 37G10; 34C07; 34C23

Keywords: Cyclicity; Limit cycle; Center; Center manifold

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1. Introduction

We consider real polynomial families of differential equations

$$\dot{x} = -y + \mathcal{F}_1(x, y, z; \mu), \quad \dot{y} = x + \mathcal{F}_2(x, y, z; \mu), \quad \dot{z} = \lambda z + \mathcal{F}_3(x, y, z; \mu) \tag{1}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ contain only nonlinear terms, under the assumption that the origin is an isolated singular point for all values of the parameters $\mu \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and where the parameter vector μ is comprised of all the admissible arbitrary coefficients in the nonlinearities of family (1). Any analytic family of differential equations on $\mathbb{R}^3$ having an isolated *Hopf singularity*, by which we mean a singularity at which the linear part possesses one real and two purely imaginary eigenvalues, can be transformed into the form (1) by means of an affine change of coordinates and a rescaling of time. The reader can consult the classical source [18] for the study of periodic orbits around Hopf points in $\mathbb{R}^n$.

It is well known that for any positive integer k family (1) has a two-dimensional C^k (local) *center manifold* W^c at the origin. But, depending on the values of the parameters μ, W^c can be nonunique and also nonanalytic (see, for instance, [2] and [23]). Nevertheless, the local dynamics of (1) near the origin restricted to any two center manifolds are C^{k-1}-conjugate and must be of either focus or center type (see [6] and [2], respectively; the latter fact can also be easily deduced from the technique developed in [5]). We say that the origin is a *center* of (1) if all the orbits on the local center manifold at the origin are periodic (and in this case the center manifold is unique and analytic); otherwise it is called a *saddle-focus*.

For any fixed choice of λ there are associated to the origin of family (1) two different sequences of polynomials $\{v_j(\lambda, \mu)\}_{j\in\mathbb{N}} \subset \mathbb{R}(\lambda)[\mu]$ and $\{\widetilde{\eta}_j(\lambda, \mu)\}_{j\in\mathbb{N}} \subset \mathbb{R}(\lambda)[\mu]$, called *Poincaré–Lyapunov quantities* and the *real focus quantities* respectively. The first sequence arises from the study of the first return map of (1) near the origin, following the ideas of [5]. The second one arises as obstructions to the existence of a certain type of formal first integral for (1) (see [10]). Both share the property that system (1) with $(\lambda, \mu) = (\lambda^*, \mu^*)$ has a center at the origin if and only if every element of the sequence vanishes at $(\lambda, \mu) = (\lambda^*, \mu^*)$. The focus quantities are much easier to work with than the Poincaré–Lyapunov quantities because their computation is algorithmic and highly efficient, since only algebraic manipulations are required, not quadratures, as in the case of the Poincaré–Lyapunov quantities.

We will prove that these two sequences of elements of $\mathbb{R}(\lambda)[\mu]$ can be related so that by means of computational algebra techniques we can in some cases bound the cyclicity of centers at the origin in (1), where roughly speaking the *cyclicity* of the singularity at the origin of system (1) with $(\lambda, \mu) = (\lambda^\dagger, \mu^\dagger)$ is the maximum number of limit cycles that can bifurcate from it under small perturbation of the parameters in (1), that is, for $(\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^p$ with $|\lambda - \lambda^\dagger| \ll 1$ and $\|(\mu - \mu^\dagger)\| \ll 1$. (The precise statement is Definition 10.)

Actually, one more limit cycle can be created if we introduce one more real parameter α and we permit in (1) perturbations of the form

$$\dot{x} = \alpha x - y + \mathcal{F}_1(x, y, z; \mu), \ \dot{y} = x + \alpha y + \mathcal{F}_2(x, y, z; \mu), \ \dot{z} = \lambda z + \mathcal{F}_3(x, y, z; \mu). \tag{2}$$

We do not count this extra limit cycle in this work.

When the origin of a specific system (1) is a saddle-focus a bound on its cyclicity can be found by reducing the problem to the study of the two-dimensional system that arises when the full system is restricted to any local center manifold W^c (see for example [17] and also

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