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On the second order derivative estimates for degenerate parabolic equations

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Abstract

We study the parabolic equation

$$
\begin{aligned}
&u_t(t,x) = a^{ij}(t)u_{x^ix^j}(t,x) + f(t,x), \quad (t,x) \in [0,T] \times \mathbf{R}^d \\
&u(0,x) = u_0(x)
\end{aligned}
\tag{0.1}
$$

with the full degeneracy of the leading coefficients, that is,

$$
(a^{ij}(t)) \geq \delta(t) I_{d\times d} \geq 0. \tag{0.2}
$$

It is well known that if f and u_0 are not smooth enough, say $f \in \mathbb{L}_p(T) := L_p([0,T]; L_p(\mathbf{R}^d))$ and $u_0 \in L_p(\mathbf{R}^d)$, then in general the solution is only in $C([0,T]; L_p(\mathbf{R}^d))$, and thus derivative estimates are not possible.

In this article we prove that $u_{xx}(t,\cdot) \in L_p(\mathbf{R}^d)$ on the set $\{t : \delta(t) > 0\}$ and

$$
\int_0^T \|u_{xx}(t)\|_{L_p}^p \delta(t)dt \leq N(d,p) \left(\int_0^T \|f(t)\|_{L_p}^p \delta^{1-p}(t)dt + \|u_0\|^p_{B_p^{2-2/p}} \right),
$$

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where $B_p^{2-2/p}$ is the Besov space of order $2-2/p$. We also prove that $u_{xx}(t,\cdot) \in L_p(\mathbf{R}^d)$ for all $t>0$ and

$$\int_0^T \|u_{xx}\|^p_{L_p(\mathbf{R}^d)}\,dt \le N\|u_0\|^p_{B_p^{2-2/(\beta p)}}, \tag{0.3}$$

if $f=0$, $\int_0^t \delta(s)ds>0$ for each $t>0$, and a certain asymptotic behavior of $\delta(t)$ holds near $t=0$ (see (1.3)). Here $\beta>0$ is the constant related to the asymptotic behavior in (1.3). For instance, if $d=1$ and $a^{11}(t)=\delta(t)=1+\sin(1/t)$, then (0.3) holds with $\beta=1$, which actually equals the maximal regularity of the heat equation $u_t=\Delta u$.

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1. Introduction

The study of degenerate elliptic and parabolic equations was started long time ago. For instance, L_2-theory for the fully degenerate elliptic and parabolic equation was developed in [13–16], and L_p-theory was introduced in [4,10]. In these articles, it is assumed that $A:=(a^{ij})\ge 0$ and it depends both on t and x. Under the such general degeneracy, the solution is only in $C([0,T];L_p)$ and can not be smoother than f and u_0. This can be easily seen by taking $A=0$ so that

$$u(t)=u(0)+\int_0^t f\,ds, \quad \forall t>0. \tag{1.1}$$

Thus, some extra conditions on the degeneracy are needed for better regularity of solutions. Especially there are many articles handling the degeneracy depending mainly on the space variable. See e.g. [17,9,7,2] (analytic methods) and [18,3] (probabilistic method). These results focused on controlling the degeneracy of $a^{ij}(x)$ near the boundary of domains and used certain weights or considered splitting the boundary with a modified Dirichlet condition.

In this article we investigate a maximal regularity of solutions under the condition that the degeneracy depends only on the time variable, that is $a^{ij}=a^{ij}(t)$. Based on a standard perturbation argument one can extend our result to the general case $a^{ij}=a^{ij}(t,x)$ (see Remark 2.9). As mentioned in the abstract above, under condition (0.2), we prove that $u_{xx}(t)\in L_p$ on $\{t: A(t)>0\}$ and

$$\|u_{xx}\|_{L_p([0,T],\delta(t)dt;L_p)} \le N(d,p)\left(\|f\|_{L_p([0,T],\delta^{1-p}(t)dt;L_p)}+\|u_0\|_{B_p^{2-2/p}}\right). \tag{1.2}$$

This might look absurd at first glance if one considers the above example (see (1.1)), but in such case (1.2) is trivial because the left hand side is zero.

To estimate u_{xx} without the help of the weight $\delta(t)dt$, we further assume that $\int_0^t \delta(s)ds>0$ for any $t>0$, $|a^{ij}(t)|\le N\delta(t)$, and for some $\beta, t_0, N_0>0$,