ARTICLE IN PRESS YJDEQ:9422

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

The Deift–Zhou steepest descent method to long-time asymptotics for the Sasa–Satsuma equation

Huan Liu *, Xianguo Geng, Bo Xue

School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, People's Republic of China

Received 23 March 2018; revised 27 May 2018

Abstract

The initial value problem for the Sasa–Satsuma equation is transformed to a 3×3 matrix Riemann–Hilbert problem with the help of the corresponding Lax pair. Two distinct factorizations of the jump matrix and a decomposition of the vector-valued function $\rho(k)$ are given, from which the long-time asymptotics for the Sasa–Satsuma equation with decaying initial data is obtained by using the nonlinear steepest descent method.

Keywords: Riemann–Hilbert problem; Sasa–Satsuma equation; Long-time asymptotics

1. Introduction

In this paper, we focus on the long-time asymptotics of the Cauchy problem for the Sasa–Satsuma equation [1], called also the higher-order nonlinear Schrödinger (NLS) equation,

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0, \tag{1}$$
$$u(x,0) = u_0(x),$$

* Corresponding author.
E-mail address: liuhuan@zzu.edu.cn (H. Liu).

https://doi.org/10.1016/j.jde.2018.07.026

where $u(x,t)$ is complex-valued and $u_0(x) \in \mathscr{S}(\mathbb{R})$ lies in the Schwartz space. Eq. (1) has some important physical background, for example, in deep water waves [2] and generally in dispersive nonlinear media [3]. Recently, this equation has caught considerable attention and been studied extensively. For instance, both the inverse scattering transform (IST) and the Hirota bilinear method have been used to obtain the N-soliton solution of equation (1) [1,4]. The Riemann–Hilbert method has been used to derive the squared eigenfunctions of Sasa–Satsuma equation [5]. Researchers also studied various properties for the Sasa–Satsuma equation such as infinite conservation laws, dark soliton solutions, nonlocal symmetries, the Painlevé property, the rogue wave spectra etc [6–9].

IST [10,11] is a major milestone in integrable systems, in fact, the inverse scattering problem usually can be written as a Riemann–Hilbert factorization problem. As is well-known, one can not solve this problem in a closed form unless in the case of reflectionless potentials. Naturally, a next issue is to investigate the asymptotic behavior for the solution. There were a number of progresses in this subject, particularly significant among them is the nonlinear steepest descent method [12] (also called Deift–Zhou method) for Riemann–Hilbert (RH) problems.

In this paper, by generalizing the nonlinear steepest descent method, we shall derive the long-time asymptotics of the initial value problem for the Sasa–Satsuma equation with a 3×3 Lax pair. There have been the asymptotics for a number of nonlinear integrable equations with 2×2 Lax pairs, for instance, KdV, NLS, mKdV, sine-Gordon, Camassa–Holm, derivative NLS etc [12–21], but there is just a little of literature about nonlinear integrable equations with 3×3 Lax pairs [25,26]. Therefore, how to analyze asymptotic behavior of nonlinear integrable equations with 3×3 Lax pairs is interesting and significant. One makes the analysis more complicated and difficult because of the integrability characterized by a higher order Lax pair. Thus the analysis here presents some novelties: (a) In Refs. [25,26], it is necessary to make use of the Fredholm integral equation to formulate the RH problem, however, it is optional, but not necessary in this context. In fact, two eigenvalues of the Lax pair for Sasa–Satsuma equation are equal, if we adopt the block notations for the matrices as in our early work [22–24], then the 3×3 matrix RH problem will be formulated directly by the eigenfunctions of the Lax pair. With these notations, although the Lax pair has the 3×3 structure, the matrices can be reduced to the 2×2 block ones, and the formalism of the Deift–Zhou method will take the (more habitual) 2×2 form. (b) The function denoted by $\delta(k)$ usually appears on the studies for asymptotic analysis by nonlinear steepest descent method. By Plemelj formula, function δ always can be solved in a closed form in the Refs. [12–21], unlike the former, $\delta(k)$ in the present paper is the solution of a matrix-valued RH problem. Since the solution of this RH problem cannot be given explicitly, we cannot directly perform scaling transformation to reduce the RH problem to a model one. Recalling that the topic of this paper is studying the asymptotic behavior of solution, we can replace function $\delta(k)$ with $\det \delta(k)$ by adding an error term.

The main result of this paper is expressed as follows:

Theorem 1. *Suppose $u(x,t)$ solves the Sasa–Satsuma equation (1) with $u_0 \in \mathscr{S}(\mathbb{R})$. Then, when $x < 0$ and $\left|\frac{x}{t}\right|$ is bounded, the solution $u(x,t)$ has the leading asymptotics*

$$u(x,t) = u_a(x,t) + O\left(c(k_0)\frac{\log t}{t}\right),$$

where

Please cite this article in press as: H. Liu et al., The Deift–Zhou steepest descent method to long-time asymptotics for the Sasa–Satsuma equation, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.07.026