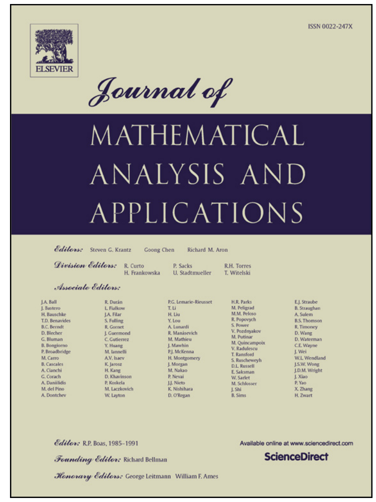

Proof of the Deficiency Index Conjecture

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Abstract. We prove that all possible values of the deficiency indices r, s of a symmetric (formally self-adjoint) differential expression M with complex coefficients which satisfy the well known inequalities are realized. Subject to these inequalities, in 1974 Kogan and Rofe-Beketov showed that all values of r,s which differ by no more than 1 can be realized, in 1978, 1979 Gilbert showed that their difference can be albitrarily large provided the order of M is large enough. In this paper we show that all values of r and s are realized.

1. Introduction

For a symmetric expression M and a weight function w, the deficiency indices d^+, d^-, are defined as the number of linear independent solutions of the equation

$$M\,y = \lambda w\,y \text{ on } J,\ \lambda \in \mathbb{C} \tag{1.1}$$

which are in $H = L^2(J, w)$. Let $\mathbb{C}$ denote the complex numbers and $\mathbb{R}$ the real numbers. The deficiency indices depend on M, w, J and λ.

Theorem 1. *For fixed* (M, w, J), $d(\lambda_1) = d(\lambda_2)$ *if* $\mathrm{Im}(\lambda_j) > 0$ *or if* $\mathrm{Im}(\lambda_j) < 0$ *for* $j = 1, 2$.

Proof. This follows from a well known abstract result. The proof given on p. 33 of Naimark's book [**12**, Theorem5, p.33] can readily be adapted to the much more general class of symmetric expressions M constructed in Section 2 below. □

Definition 1. *Let* $d^+ = d^+(M, w, J) = d^+(i)$ *and* $d^- = d^-(M, w, J) = d^-(-i)$.

What are the possible values of d^+, d^- ? This is the question we answer in this paper.

It is clear from the definition that

$$0 \leq d^+, d^- \leq n \tag{1.2}$$

where n is the order of M since equation (1.1) cannot have more than n linearly independent solutions.

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