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# Some character generating functions on Banach algebras

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ABSTRACT

We consider a multiplicative variation on the classical Kowalski–Słodkowski Theorem which identifies the characters among the collection of all functionals on a Banach algebra $A$. In particular we show that, if $A$ is a $C^*$-algebra, and if $\phi : A \mapsto \mathbb{C}$ is a continuous function satisfying $\phi(\mathbf{1}) = 1$ and $\phi(x)\phi(y) \in \sigma(xy)$ for all $x, y \in A$ (where $\sigma$ denotes the spectrum), then $\phi$ generates a corresponding character $\psi_\phi$ on $A$ which coincides with $\phi$ on the principal component of the invertible group of $A$. We also show that, if $A$ is any Banach algebra whose elements have totally disconnected spectra, then, under the aforementioned conditions, $\phi$ is always a character.



## 1. Introduction

In this paper $A$ will always be a complex and unital Banach algebra, with the unit denoted by $\mathbf{1}$. The invertible group of $A$ will be denoted by $G(A)$, and the connected component of $G(A)$ containing $\mathbf{1}$, by $G_{\mathbf{1}}(A)$. It is well known (see for instance [1, Theorem 3.3.7]) that

$$G_{\mathbf{1}}(A) = \{e^{x_1} \cdots e^{x_k} : k \in \mathbb{N},\ x_j \in A\}. \tag{1.1}$$

If $x \in A$ then the spectrum of $x$ is the (necessarily non-empty and compact) set $\sigma(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\}$. A character of $A$ is, by definition, a linear functional $\chi : A \to \mathbb{C}$ which is simultaneously multiplicative i.e. $\chi(xy) = \chi(x)\chi(y)$ holds for all $x, y \in A$. Depending on the specific algebra, or class of algebras, characters may or may not exist. One immediately recalls the famous result of Gleason, Kahane, and Żelazko [3,5,12] which identifies the characters among the dual space members of $A$ via a spectral condition:

**Theorem 1.1** *(Gleason–Kahane–Żelazko). Let $A$ be a Banach algebra. Then $\phi \in A'$, the dual of $A$, is a character of $A$ if and only if $\phi(x) \in \sigma(x)$ for each $x \in A$.*

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A perhaps lesser known, but stronger result, due to Kowalski and Słodkowski [6], identifies the characters among all complex-valued functions on $A$ via a spectral condition:

**Theorem 1.2** *(Kowalski–Słodkowski). Let $A$ be a Banach algebra. Then a function $\phi : A \to \mathbb{C}$ is a character of $A$ if and only if $\phi$ satisfies*

(i) $\phi(0) = 0$,
(ii) $\phi(x) - \phi(y) \in \sigma(x - y)$ *for every* $x, y \in A$.

**Remark 1.3.** It is easy to see that the Kowalski–Słodkowski Theorem can be more economically formulated as

$$\phi(x) + \phi(y) \in \sigma(x + y) \text{ for every } x, y \in A \Leftrightarrow \phi \text{ is a character of } A.$$

One is now naturally led to ask whether there exist "multiplicative" versions of Theorem 1.1 and Theorem 1.2. That is, under what conditions is a function with multiplicative properties, perhaps involving the spectrum, a character? The problem seems thorny, but some positive results were obtained in [7] and [11].

**Theorem 1.4** *([11, Corollary 2.2]). Let $A$ be a Banach algebra. Then a multiplicative function $\phi : A \to \mathbb{C}$ satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$ is a character if and only if for each $x \in A$ the map*

$$\lambda \mapsto |\phi(x - \lambda \mathbf{1}) + \lambda| \tag{1.2}$$

*is subharmonic on $\mathbb{C}$.*

One may show ([11, Theorem 2.1]) that the subharmonic condition on (1.2) in Theorem 1.4 can be replaced with the requirement that $\lambda \mapsto \phi(x - \lambda\mathbf{1})$ is an entire function for each $x \in A$.

**Theorem 1.5** *(Maouche). Let $A$ be a Banach algebra, and let $\phi : A \to \mathbb{C}$ be a multiplicative function satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$. Then, corresponding to $\phi$, there exists a unique character on $A$ which agrees with $\phi$ on $G_{\mathbf{1}}(A)$.*

Among a number of results for $C^\star$-algebras (on the above-mentioned topic), it is further shown, in [11], that for the particular case of von Neumann algebras, a continuous multiplicative function with values $\phi(x) \in \sigma(x)$, for each $x \in A$, is always a character.

The current paper is motivated by *multiplicatively* spectrum preserver problems which were studied already in 1997 by B. Aupetit in [2, Theorem 3.5, p. 74], and later also in [4,8–10], as well as the Kowalski–Słodkowski Theorem. We shall consider a function $\phi : A \to \mathbb{C}$ (not assumed to be linear or multiplicative) satisfying the following conditions:

(P1) $\phi(x)\phi(y) \in \sigma(xy)$ for all $x, y \in A$,
(P2) $\phi(\mathbf{1}) = 1$,
(P3) $\phi$ is continuous on $A$.

In [4] Hatori et al. show that a multiplicative Kowalski–Słodkowski Theorem is generally not possible. In particular, even for commutative $C^*$-algebras, the conditions (P1)–(P2) are not enough to guarantee that $\phi$ is a character (see also [11, p. 56] and [7, pp. 44–45]). What we want to show here is that some positive results can be obtained (with (P3) added to the list) for at least two classes of Banach algebras, one of which is general $C^*$-algebras. Our proofs rely on the Lie–Trotter Formula, stated below, and the additive