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Quasinilpotent operators and non-Euclidean metrics

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ABSTRACT

The *power set* $\Lambda(V)$ of quasinilpotent operator V on a Hilbert space $\mathcal{H}$ is defined in [4] to study the singularity of the non-Euclidean metrics $\|(V-z)^{-1}x\|^2 dz \otimes d\bar{z}$ at $\sigma(V) = \{0\}$, and it is shown that if $\Lambda(V)$ contains more than one point then V has a nontrivial hyperinvariant subspace. This paper first proves that the Volterra integral operator on the classical Hardy–Hilbert space has singleton power set, thus answering a question raised in [4]. Then, it studies the length of circles under the metrics and its connection with power set. In particular, it determines the maximal length of the unit circle with respect to the change of x in the metrics. Moreover, it shows that an extremal value integral equation is able to detect representing functions for all the invariant subspaces of the Volterra operator.

0. Introduction

Consider a densely defined (possibly unbounded) linear operator A acting on a Hilbert space $\mathcal{H}$. A complex number λ is said to be regular for A if $A - \lambda I$ has a bounded inverse. The set of regular points is denoted by $\rho(A)$ and is called the resolvent set for A. The spectrum of A is $\sigma(A) := \mathbb{C} \setminus \rho(A)$. It is well-known that $\rho(A)$ is an open set. According to Banach inverse mapping theorem, if A is bounded then $\sigma(A)$ coincides with the conventional spectrum since $(A - zI)^{-1}$ is linear and bounded whenever $z \in \rho(A)$.

Let $L(\mathcal{H})$ be the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. The $L(\mathcal{H})$-valued Maurer–Cartan type $(1,0)$-form

$$\omega_A(z) = (A-z)^{-1} d(A-z) = -(A-z)^{-1} dz, \qquad z \in \rho(A),$$

plays a central role in classical functional calculus. For instance, when A is bounded and γ is a piece-wise smooth simple closed curve enclosing $\sigma(A)$, then for every f analytic on a neighborhood D that contains $\sigma(A)$ and γ we can write

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$$f(A) = \frac{-1}{2\pi i} \int_{\gamma} f(z)\omega_A(z). \tag{0.1}$$

The adjoint of $\omega_A(z)$ is the $L(\mathcal{H})$-valued $(0,1)$-form $\omega_A^*(z) = -(A^* - \bar{z})^{-1} d\bar{z}$. In [4], the *fundamental form* for A is defined as

$$\Omega_A = -\omega_A^* \wedge \omega_A = (A^* - \bar{z})^{-1}(A - z)^{-1} dz \wedge d\bar{z}. \tag{0.2}$$

For every fixed $x \in \mathcal{H}$ with $\|x\| = 1$, the vector state ϕ_x on $L(\mathcal{H})$ is the positive linear functional defined by $\phi_x(T) = \langle Tx, x\rangle,\ T \in L(\mathcal{H})$. Then on the resolvent set $\rho(A)$, we have

$$\phi_x(\Omega_A) = -\phi_x(\omega_A^* \wedge \omega_A) = \|(A - zI)^{-1}x\|^2 dz \wedge d\bar{z}. \tag{0.3}$$

Since $g_x(z) = \|(A - zI)^{-1}x\|^2 > 0$ everywhere on $\rho(A)$, $g_x(z)dz \otimes d\bar{z}$ defines a Hermitian metric on $\rho(A)$. Unlike the usual Euclidean metric on $\rho(A)$, this metric contains information about A and its action on x, and moreover, it may have singularities at points in $\sigma(A)$. This new metric defined on the resolvent set is introduced in [4]. Some relevant studies can be found in earlier papers [5], [7], [10] and the references therein. A particularly interesting case is when A is quasinilpotent (see, e.g. [11]), i.e., when $\sigma(A) = \{0\}$, in which case the metric lives on the punctured complex plane and it blows up as z tends to 0. In order to measure how quickly the metric $g_x(z)$ diverges to infinity as z tends to 0, the following definition was made in [4]:

$$k_x = \limsup_{|z|\to 0} \frac{\log \|(A - z)^{-1}x\|}{\log \|(A - z)^{-1}\|},\ x \neq 0, \tag{0.4}$$

and further defined the power set of A as $\Lambda(A) := \{k_x :\ x \in \mathcal{H},\ x \neq 0\}$. It holds that $k_{rf} = k_f$ for any $r \neq 0$ and $k_{f+g} \leq k_f + k_g$ for any $f,\ g \in \mathcal{H}$.

For $A \in L(\mathcal{H})$, the commutant of A, denoted by $\{A\}'$, is the algebra of all operators T in $L(\mathcal{H})$ such that $TA = AT$. A closed subspace $M \subset \mathcal{H}$ is called a nontrivial hyperinvariant subspace for A if $\{0\} \neq M \neq \mathcal{H}$ and $TM \subset M$ for each $T \in \{A\}'$. The hyperinvariant subspace problem is the question of whether every operator in $L(\mathcal{H})$ has a nontrivial hyperinvariant subspace. This problem remains open, particularly so for quasinilpotent operators in $L(\mathcal{H})$. We refer readers to [2,6,8] and the references therein for more information. It has been shown in [4] that the power set $\Lambda(A)$ is a subset of the interval $[0,1]$, and it has a close link with A's lattice of hyperinvariant subspaces. In fact, for $0 \leq \tau \leq 1$ if one sets

$$M_\tau = \{x \in \mathcal{H} : x = 0 \ \text{ or } k_x \leq \tau\}$$

then [4] obtained the following

Theorem 0.1. *For each $0 \leq \tau \leq 1$, the space M_τ is hyperinvariant for A.*

As an immediate consequence, if $\Lambda(A)$ contains more than one point then A has a nontrivial hyperinvariant subspace. The following question was raised in [4].

Question. Does there exist a quasinilpotent operator whose power set is singleton?

First, this paper will answer the question positively. It will prove that if V is the Volterra integral operator on the Hardy space $H^2(\mathbb{D})$, then $\Lambda(V) = \{1\}$. Then, it will study the length of circle with respect to the metric $\|(V - z)^{-1}x\|^2 dz \otimes d\bar{z}$ and determine its maximal value. Furthermore, it will show that an extremal value integral equation regarding the length is able to detect all the generating functions of the invariant subspaces of V.

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