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On linear maps preserving complex symmetry

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ABSTRACT

An operator T on a complex Hilbert space $\mathcal{H}$ is said to be complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $CTC = T^*$. This paper is devoted to describing which linear maps leave the class of complex symmetric operators invariant. Complete characterizations are obtained for several classes of linear maps, including similarity transformations, surjective linear isometries, multiplication operators and certain completely positive maps.

1. Introduction

Throughout this paper, we denote by $\mathcal{H}$ a complex separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called a *complex symmetric operator* (CSO, for short) if there exists a conjugation C on $\mathcal{H}$ so that $CTC = T^*$. A conjugation on $\mathcal{H}$ means a conjugate-linear, isometric involution on $\mathcal{H}$. We remark that an operator T is complex symmetric precisely when there is an orthonormal basis of $\mathcal{H}$ with respect to which T has a complex symmetric matrix representation (see [3, Lemma 2.16]).

Complex symmetric matrices have been studied for many years, and have many motivations in complex analysis, differential equations, function theory and other areas. The general study of complex symmetric operators, especially in the infinite dimensional case, was initiated by S. Garcia, M. Putinar and W. Wogen (see [4,5,7,8] for references), and has recently received much attention.

We remark that complex symmetry is a well-hidden symmetry, since usually it is difficult to determine whether a given operator is complex symmetric even on finite dimensional spaces (see [1,2,16]). Many research papers are devoted to describing concrete CSOs (see [8,7,12,13,17,18] for references).

It is the purpose of this paper to enrich the theory of complex symmetric operators by studying the following linear preserver problem.

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Problem 1.1. Characterize those linear maps ϕ on $\mathcal{B}(\mathcal{H})$ that satisfy

$$X \text{ is a CSO} \Longrightarrow \phi(X) \text{ is a CSO.} \tag{1}$$

We say that a linear map ϕ on $\mathcal{B}(\mathcal{H})$ preserves complex symmetry if it satisfies (1).

As one of the most active areas in matrix theory, the linear preserver problem concerns the characterizations of linear transformations on matrix spaces that preserve certain properties or leave certain functions, relations invariant. The reader is referred to [10,14] for more about the subject. Also there have been growing interest in linear preserver problems on operator algebras. The present paper lies in the category and is devoted to describing linear preservers of complex symmetry on operator spaces.

One can immediately construct some linear maps on $\mathcal{B}(\mathcal{H})$ preserving complex symmetry.

Example 1.2. If $U \in \mathcal{B}(\mathcal{H})$ is unitary, then the unitary transformation $\phi : X \mapsto UXU^*$ preserves complex symmetry. In fact, X is a CSO if and only if so is $\phi(X)$.

Example 1.3. Let C be a conjugation on $\mathcal{H}$ and ϕ be the linear map on $\mathcal{B}(\mathcal{H})$ defined as $\phi(X) = CX^*C$ for $X \in \mathcal{B}(\mathcal{H})$. It is easy to check that ϕ is a surjective linear isometry and also a linear preserver of complex symmetry. We remark that ϕ is in fact a transpose transformation of operators with respect to some orthonormal basis (ONB, for short) of $\mathcal{H}$. Since C is a conjugation, by [4, Lemma 1], there exists an ONB $\{e_n\}$ of $\mathcal{H}$ such that $Ce_n = e_n$. For i, j, we have

$$\begin{aligned}\langle \phi(X)e_j, e_i\rangle &= \langle CX^*Ce_j, e_i\rangle = \langle CX^*e_j, e_i\rangle \\ &= \langle Ce_i, X^*e_j\rangle = \langle e_i, X^*e_j\rangle = \langle Xe_i, e_j\rangle.\end{aligned}$$

This shows that the matrix representation of $\phi(X)$ is the transpose of that of X with respect to $\{e_n\}$.

Example 1.4. Let C be a conjugation on $\mathcal{H}$ and ϕ be the linear map on $\mathcal{B}(\mathcal{H})$ defined as

$$\phi(X) = \frac{CX^*C + X}{2}, \quad \forall X \in \mathcal{B}(\mathcal{H}).$$

It is easy to check that $\phi(X)$ is complex symmetric relative to the conjugation C for all $X \in \mathcal{B}(\mathcal{H})$. Thus ϕ is a linear preserver of complex symmetry. In addition ϕ is idempotent, that is, $\phi \circ \phi = \phi$.

Example 1.5. Let C be a conjugation on $\mathcal{H}$ with $\dim \mathcal{H} = \infty$ and $U : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ be a unitary operator. For $X \in \mathcal{B}(\mathcal{H})$, define $\phi(X) = U(X \oplus CX^*C)U^*$. Note that unitary transformations preserve complex symmetry and an operator of form $X \oplus CX^*C$ is always complex symmetric; so ϕ is a linear preserver of complex symmetry. Also one can check that ϕ is isometric and not surjective.

Indeed, there are too many linear maps on $\mathcal{B}(\mathcal{H})$ preserving complex symmetry. In the present paper we concentrate on some special linear maps (including similarity transformations, surjective linear isometries and multiplication operators) and solve Problem 1.1 for these linear maps. Also it is determined when completely positive maps induced by single operators preserve complex symmetry (Theorem 4.1). As one shall see, the proofs of these results depend on many matrix techniques. Our results suggest that it is not an easy task to classify linear preservers of complex symmetry.

Since each operator acting on Hilbert spaces of dimension not greater than 2 is complex symmetric (see [8, Theorem 1]), we need only consider Problem 1.1 in the case that $\dim \mathcal{H} \geq 3$. So, in the sequel, $\mathcal{H}$ is always assumed to satisfy $\dim \mathcal{H} \geq 3$.