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Remarks on energy methods for structure-preserving finite difference schemes – Small data global existence and unconditional error estimate

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ABSTRACT

In the previous article (Yoshikawa, 2017), the author proposes the energy method for structure-preserving finite difference schemes, which enable us to show global existence and uniqueness of solution for the schemes and error estimates. In this article, we give two extended remarks of the methods. One is related to the small data global existence results for schemes of which energy is not necessarily bounded from below. The other is an unconditional error estimate which holds globally in time and without smallness condition for split sizes. These results can be shown due to the structure-preserving property.

1. Introduction

The aim of this article is to give two extended remarks on the methods proposed in [16]. One is related to the *small data global existence* result for structure-preserving numerical schemes of which energy is not necessarily bounded from below. The other is concerned with an error estimate between strict and approximate solutions. We shall give the error estimate which holds globally in time and does not require smallness assumption for split sizes. Here we call it the *unconditional error estimate*.

The common keyword of these is “global-in-time”. The global property is one of main topics in not only partial differential equations (PDEs) but also numerical analysis. Indeed, we can find the keywords in many literatures (see e.g. [1,9,13], etc). To obtain the global property, energy structures often play an essential role as a priori estimates. Multiplying an equation of motion by velocity, we derive an energy conservation law if the equation has no external force term. In a similar way we can derive a various a priori estimates for other equations by multiplying unknowns and their derivatives. These quantities are generically called energies in PDEs even if these are not strictly energies from physical viewpoint, and to bring mathematical properties out by the multiplication is called the *energy method* generally. For example, it is a standard procedure that we construct a local-in-time solution in an energy class and extend it to global-in-time solution with the help of a priori estimate derived by the energy structure (see e.g. [2,13]). The energy class means the function space naturally defined from the energy. Although the energy method can be often applied to the proof of existence of solution as mentioned above, we may also show uniqueness and continuous dependence.

We expect to obtain similar results for numerical schemes if these have the global property such as energy structure. However, we remark that a loss of global property often occurs when we construct numerical schemes from PDEs. The

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most popular example is the explicit scheme for a heat equation, which is numerically unstable without stability conditions for splitting sizes. From the maximum principle, the strict solution has to be globally positive under positive initial data. On the other hand, the solution for the explicit scheme may be negative values, which implies the loss of global property. To obtain global stable numerical schemes, structure-preserving numerical methods are one of effective strategies. We call numerical schemes which inherit the energy structure for the differential equations *structure-preserving numerical schemes*. Although there are many kinds of structures such as symplectic structure (see e.g. [8,14,15]) and energy structure etc., in this article the preserved structure is physical one such as energy conservation laws and entropy increasing laws. There are many results to study the structure-preserving numerical schemes (see e.g. [3–7,11] and reference therein).

When we study numerical analysis for evolution equations especially in nonlinear cases, the structure-preserving numerical methods provide us with a lot of benefit. Indeed, in the result [16], translating the context of the energy method into structure-preserving numerical schemes, the author proves the large data global existence and uniqueness of solution for the scheme, where the solution for the original equation satisfies conservation or dissipation of energy which is bounded from below. In general, when the energy is not bounded from below, existence of global solution with arbitrarily initial data cannot be expected because of a lack of boundedness for solution. However, it is well-known that existence of global solution may be shown under the smallness assumption for initial data (see e.g. [13]). We shall transplant the argument for PDEs into structure-preserving numerical schemes. Moreover, in [16] analogizing to the continuous dependence, an error estimate between strict and approximate solutions is shown, where the error estimate requires some smallness assumption for splitting sizes and in addition holds in bounded time interval. The restriction for time interval is required by usage of the discrete Gronwall inequality. Hence, for the equation which possesses some dissipativity we can remove the exponentially growth term with respect to the size of time interval.

The rest of this article is organized as follows. In Section 2, we give main topics on small data global existence (Theorem 2.4) and unconditional error estimate (Theorem 2.7) by using simple ordinary differential equation (ODE) as an example. These theorems can be applied to nonlinear PDE, in Section 3, we shall introduce the application.

2. Energy method

In this section, we introduce main topics of this article by using a simple example from ODE. Throughout this paper, we denote a positive constant by C, which may change line to line. Moreover, we shall use specific positive constants such as C_0, C_1, C_2, C_l, C_S and C_{Sd}. In this section we first set up notation, terminology and several fundamental lemmas which will be used later. Afterward, we explain main topics of this paper.

We denote partial differential operators with respect to space variable x and time variable t by ∂_x and ∂_t, and similarly we define the differential operators with respect to ξ, η by ∂_ξ, ∂_η, respectively. In particular, in the case of single variable function we may also denote the derivatives such as F', F'' and $F''{}'$.

2.1. Several tools for difference quotient

Let Ω be a domain in $\mathbb{R}$. For $F \in C^1(\Omega)$ and $\xi, \eta \in \Omega$ we define *difference quotient* of F at (ξ, η) by

$$\frac{\partial F}{\partial(\xi,\eta)} := \begin{cases} \frac{F(\xi)-F(\eta)}{\xi-\eta}, & \xi \neq \eta, \\ F'(\eta), & \xi = \eta. \end{cases}$$

It often appears naturally in structure-preserving numerical schemes (see e.g. [7]). For example, in the case that $F(\xi) = \frac{1}{p+1}\xi^p$ for $p \in \mathbb{N}$, its difference quotient is given as $\frac{\partial F}{\partial(\xi,\eta)} = \frac{1}{p+1}\sum_{j=0}^{p}\xi^j\eta^{p-j}$.

Subtraction between two difference quotients often appears in both proofs of existence of solution and error estimate. In order to treat these calculations simply and systematically, we define $\overline{F}''(\xi,\widetilde{\xi};\eta,\widetilde{\eta})$ for $F \in C^2$ by

$$\begin{aligned} \overline{F}''(\xi,\widetilde{\xi};\eta,\widetilde{\eta}) &:= \frac{\partial}{\partial(\xi,\widetilde{\xi})}\left(\frac{\partial F}{\partial(\cdot,\eta)} + \frac{\partial F}{\partial(\cdot,\widetilde{\eta})}\right) \\ &= \begin{cases} \frac{1}{\xi-\widetilde{\xi}}\left\{\left(\frac{\partial F}{\partial(\xi,\eta)} + \frac{\partial F}{\partial(\xi,\widetilde{\eta})}\right) - \left(\frac{\partial F}{\partial(\widetilde{\xi},\eta)} + \frac{\partial F}{\partial(\widetilde{\xi},\widetilde{\eta})}\right)\right\}, & \xi \neq \widetilde{\xi}, \\ \partial_\xi\left(\frac{\partial F}{\partial(\xi,\eta)} + \frac{\partial F}{\partial(\xi,\widetilde{\eta})}\right)\Big|_{\xi=\widetilde{\xi}}, & \xi = \widetilde{\xi}, \end{cases} \end{aligned}$$

which is a kind of 2nd order difference quotient. For example, $\overline{F}''$ in the case $F(\xi) = \frac{1}{p+1}\xi^{p+1}$ is

$$\overline{F}''(\xi,\widetilde{\xi};\eta,\widetilde{\eta}) = \frac{1}{p+1}\sum_{j=1}^{p}\left\{\left(\eta^{p-j}+\widetilde{\eta}^{p-j}\right)\sum_{k=0}^{j-1}\xi^k\widetilde{\xi}^{j-1-k}\right\}.$$

It satisfies the following properties.