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Viability for stochastic functional differential equations in Hilbert spaces driven by fractional Brownian motion ☆

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ABSTRACT

In this paper, we consider a class of stochastic functional differential equations in Hilbert spaces driven by a fractional Brownian motion with Hurst parameter $1/2 < H < 1$. By using pathwise approach, we prove a global existence and uniqueness result of the mild solution for the equations considered under some local Lipschitz conditions. Subsequently, by establishing some new estimates, we also prove some viability results to the stochastic systems under investigation.

1. Introduction

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $B^H = \{B^H(t), t \geq 0\}$ with covariance function

$$R_H(t, s) = \mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

In particular, if $H = \frac{1}{2}$, B^H is a Brownian motion. It is well known that the fBm B^H neither is a semimartingale nor a Markov process if $H \neq 1/2$. So we can not use the classical Itô theory to construct the stochastic calculus with respect to the fBm.

However, the fBm $B^H(t)$ has α-order Hölder continuous path for all $\alpha \in (0, H)$. The pathwise Riemann–Stieltjes integral $\int_0^T u_s dB^H(s)$ which exists if the stochastic process $(u_t)_{t \in [0, T]}$ has continuous paths of order $\alpha > 1 - H$, is a consequence of the result of Young [20]. Zähle in [21] introduced a generalized Stieltjes integral using the techniques of fractional calculus. The integral is expressed in terms of fractional derivative operators and it coincides with the Riemann–Stieltjes integral $\int_0^T f dg$ when the functions f and g are Hölder continuous of order α and β, respectively, with $\alpha + \beta > 1$. Using the pathwise integral approach Ferrante and Rovira [9] obtained the existence of the solution to stochastic differential equations with the constant delay driven by fBm, Ferrante and Rovira [10] discussed the convergence of delay differential equations driven by fBm and Boufoussi and Hajji [3] proved the the existence and uniqueness of the solution to stochastic functional differential equations driven by fBm.

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Following the pathwise Riemann–Stieltjes integral approach, Ciotir and Răşcanu [7] proved a viability result for the following stochastic differential equations driven by fBm with Hurst parameter $1/2 < H < 1$

$$x(t) = x(0) + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dB^H(s), \quad t \in [0, T],$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are continuous functions. The viability property has been extensively studied for deterministic differential equations and inclusions since the pioneering work in Nagumo [15] and is still one of the active directions of differential equations, see [5,6,8,16]. The viability property in a stochastic framework was explored first by Aubin and Da Prato [1] (see also [2,11–13]) with defining a suitable Bouligand's stochastic tangent cone which generalizes the cone used in the study of the viability property for deterministic systems.

Motivated by the above discussions, in this paper we will discuss the viable problem of the following stochastic functional differential equations in Hilbert space driven by fractional Brownian motion with Hurst parameter $1/2 < H < 1$

$$\begin{cases} dx(t) = [Ax(t) + F(t, x_t)]dt + G(t, x_t)dB^H(t), & t \in [0, T], \\ x_0 = \phi \in C_r, \end{cases} \tag{1.1}$$

where A is infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space $\mathbb{H}$; B^H is a fractional Brownian motions with Hurst parameter $H > \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$; $x_t \in C_r = C_r([-r, 0]; \mathbb{H})$ denote the function defined by $x_t(\theta) = x(t + \theta)$, $\forall \theta \in [-r, 0]$; $F, G : [0, T] \times C_r \to \mathbb{H}$ are appropriate functions, while $\phi : [-r, 0] \to \mathbb{H}$ is a smooth function. Equation (1.1) is understood in the so-called mild sense, i.e., we say that a continuous process $x : [-r, T] \to \mathbb{H}$ is a mild solution of equation on $[-r, T]$ if x satisfies:

$$\begin{cases} x(t) = S(t)\phi(0) + \int_0^t S(t-s)F(s, x_s)ds + \int_0^t G(s, x_s)dB^H(s), & t \in [0, T], \\ x_0 = \phi \in C_r. \end{cases} \tag{1.2}$$

In our paper, using the Riemann–Stieltjes integral in the sense of Zähle [21]. We will first prove the existence and uniqueness of a mild solution to (1.1) under some local Lipschitz conditions. Then, by developing some new estimates, we will obtain the sufficient and necessary condition that a closed subset be a viable domain of equation.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we give some useful estimates. Section 4 is devote to obtain existence and uniqueness result of the mild solution for Eq. (1.1). In Section 5, we obtain the sufficient and necessary condition for a closed subset be a viable domain of Eq. (1.1).

2. Preliminaries

2.1. Assumptions and notations

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space and let us consider two fixed real numbers $T > 0$ and $r > 0$, we denote by $C_r = C([-r, 0], \mathbb{H})$ the space of all continuous functions from $[-r, 0]$ to $\mathbb{H}$ equipped with the norm

$$\|Z\|_{C_r} = \sup_{-r\leq s\leq 0} \|Z(s)\|.$$

If we consider a function $x \in C([-r, 0], \mathbb{H})$, for each $t \in [0, T]$ we will denote by $x_t \in C_r$ the function defined by $x_t(s) = x(t + s) \forall s \in [-r, 0]$.

Given real numbers $a < b$ and $\mu \in (0, 1]$, we denote by $C^\mu([a, b]; \mathbb{H})$ the space of μ-Hölder continuous functions $f : [a, b] \to \mathbb{H}$, equipped with the norm

$$\|f\|_{\mu;[a,b]} := \|f\|_{\infty;[a,b]} + \sup_{a\leq s\leq r\leq b} \frac{\|f(s) - f(r)\|}{(s - r)^\mu} < \infty,$$

where $\|f\|_{\infty;[a,b]} := \sup_{s\in[a,b]} \|f(s)\|$. An equivalent norm can be defined by

$$\|f\|_{\mu,\lambda;[a,b]} := \sup_{s\in[a,b]} e^{-\lambda s}\|f(s)\| + \sup_{a\leq r\leq s\leq b} e^{-\lambda r} \frac{\|f(s) - f(r)\|}{(s - r)^\mu}$$

for any $\lambda \geq 0$.

Fixed a parameter $0 < \alpha < 1/2$, let us denote by $\widetilde{W}^{1-\alpha,\infty}(a, b; \mathbb{R})$ the space of continuous functions $g : [a, b] \to \mathbb{R}$ such that

$$\|g\|_{\widetilde{W}^{1-\alpha,\infty}(a,b;\mathbb{R})} := |g(a)| + \sup_{a\leq s<r\leq b} \left(\frac{|g(r) - g(s)|}{(r - s)^{1-\alpha}} + \int_s^r \frac{|g(y) - g(s)|}{(y - s)^{2-\alpha}} dy \right) ds < \infty.$$

Clearly,

$$C^{1-\alpha+\varepsilon}(a, b; \mathbb{R}) \subset \widetilde{W}^{1-\alpha,\infty}(a, b; \mathbb{R}) \subset C^{1-\alpha}(a, b; \mathbb{R}), \quad \forall \varepsilon > 0.$$