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A new perspective to the solution and creation of zero sum matrix game with matrix norms

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ABSTRACT

We present a novel approach to solve and create a two person zero sum matrix game by using matrix norms. Especially, we show how to obtain approximated game value for any zero sum matrix game without solving any equations using our approaches. We firstly, give the results of the lemmas for the game value depend on the matrix norms of the payoff matrix and some constants k containing the game value v. Then, we introduce row-wise and column-wise induced matrix for the payoff matrix. Moreover, we improve our approaches and present some new theorems for the game value to obtain some inequalities which depend on only the $1 - norm$ and $\infty - norm$ of the payoff matrix. Furthermore, we state the min–max theorem for p_{max} and p_{min} which are the maximum and minimum elements of the mixed strategy set, respectively. Finally, we illustrate and show the consistency of our approaches with some test examples. To the best of our knowledge, this is the first study in the literature that is used the matrix norms in game theory.

1. Introduction

The game theory may be defined as a mathematical decision theory between players in a competitive environment [1]. The theory of game is established by von Neumann's work in the early 20th century. The proof of minimax theorem accelerated the development in this new field [2]. In 1944, John von Neumann and Oskar Morgenstern published a work named as Theory of Games and Economic Behavior and presented the basic principles of the game theory [3]. In the present, the game theory is an essential part of economy and mathematical finance. Especially, it is a useful tool to analyze the financial problems. However, the usage of the theory is not limited only with financial problems. The game theory also has a wide application area in real life problems. The main subject of the theory is to find the optimal choices from the set of participant's strategies in a competitive situation such as chess, military defense (or attack) problems, criminal cases, social problems, international agreements and so on. For example, in 2004, Bauch and Earn explained how the self-interest of an individual may affect the complete eradication of a vaccine-preventable disease. They analyze this problem with the formal game theoretical approaches [4]. In 2015, Egorov and Sonin analyzed the battle for throne by using game theory in [5]. D'Orsogna and Perc reviewed the mathematical models of crime where statistical physics, game theory etc., are used in an effort to figure out the various aspects of crime to determine probable prevention and amelioration strategies in the same year [6]. In 2016, Perc provided a view on the models of human cooperation from the statistical physics perspective. He presented a null model for human cooperations based on evolutionary game theory [7]. In addition, Wang and his

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collaborators comprehensively reviewed the developmental arc of theoretical epidemiology and the paradigm of digital epidemiology. They also showed the game theoretical approach to vaccination behavior with the analogy of prisoner's dilemma in 2016 [8].

In 2017, Kose et al. used the game theory combining with geographical information systems to answer a military decision problem in [9]. Wang et al. modeled the interaction of a MIMI radar and a jammer as a two person zero sum game in their paper [10]. In the same year, Wang et al. studied on the anonymous and onymous pairwise interactions among individuals in terms of the repeated Prisoner's Dilemma. In their result, they showed the strong relationships between onymity and the cooperation frequency, and the median payoff per round regarding to anonymity. Moreover, they demonstrated the correlation between player's ranks and their choice of strategies [11]. In another paper, Shen et al., in 2017, used the prisoner's dilemma game to describe the social dilemma of pairwise interactions of players and modeled the structured population on a square lattice. They emphasized the importance of the role of vertex weight with a simple structure of co-evolution model for the evolution of cooperation [12]. Arfanuzzaman et al. used the zero sum game approaches in order to make analysis for reviewing transboundary water policies, sustainable water resource management and water distributing system among countries [13]. Perc et al., in 2017, comprehensively investigated human cooperation and review the experimental and theoretical researches. They also examined the generalization of the pairwise prisoner's dilemma as the public goods game [15].

In 2018, Shen et al. investigated the behaviors of aspiration level, which affects the evolution of cooperation, with respect to the spatial prisoner's dilemma game [14]. In the same year, Yolmeh and Gursoy improved a game theoretical model for scheduling problem of security teams to patrol an urban mass transit rail network. In this study, they modeled the problem as a noncooperative simultaneous move zero sum game between the participants, the terrorist and the patrols [16]. Almost all these study rely on the zero sum games. Some of them provide a very clear usage of the game theory in the literature.

The most of the solutions of the zero sum games require the usage of the linear programming methods or other package programs. Let us consider the simplest form of a zero sum game, for an example, assume that we have 3×3 zero sum matrix game as follows:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

with $S_1 = \{p_1, p_2, p_3\}$ and $S_2 = \{q_1, q_2, q_3\}$ which are the mixed strategy sets for the row and column players, respectively. The solution of this game v is obtained by solving the optimization problems below.

For the row player, maximize v:

$$\begin{aligned} ap_1 + dp_2 + gp_3 - v &\geq 0 \\ bp_1 + ep_2 + hp_3 - v &\geq 0 \\ cp_1 + fp_2 + kp_3 - v &\geq 0 \end{aligned}$$

For the column player, minimize v:

$$\begin{aligned} aq_1 + bq_2 + cq_3 - v &\geq 0 \\ dq_1 + eq_2 + fq_3 - v &\geq 0 \\ gq_1 + hq_2 + kq_3 - v &\geq 0 \end{aligned}$$

It is clear that we need to solve two optimization problems with three inequalities for each system even in the simplest form. However, solution of these problems will be tedious for the bigger size zero sum matrix games. Therefore, in this study, we develop a practical method to provide convenience to achieve an approximated solution faster with the matrix norms of the payoff matrix. In the approach, we use $1-norm$ and $\infty-norm$ of the payoff matrix due to the easy evaluation of them. Thus, we succeed to obtain approximated solution for a two person zero sum matrix game without solving any equations by our novel method. We also present a methodology for creating the two person zero sum matrix games by using the matrix norms in this paper. Therefore, we state some conditions and present a new perspective to game creation and solution procedure. To the best of our knowledge, the matrix norms are not used in game theory yet. On the other hand, in 2018, Özkaya examine the roles of matrix norms in the game theory in his thesis [17]. Furthermore, İzgi and Özkaya extend the method for the zero sum matrix games to the nonzero sum bimatrix games in [18].

The remainder of the paper is organized as follows. In Section 2, we focus on the game value so that we state the lemmas and theorems about it. Then, we introduce row-wise and column-wise induced matrix for the payoff matrix and improve our theorems which depend on only $1-norm$ and $\infty-norm$. In Section 3, we present some theorems for the maximum and minimum elements of the mixed strategy set. Moreover, we give the min-max theorem for these strategies and prove it. In Section 4, we illustrate our methods with some test examples. We also show the consistency of our approaches by comparing our results with the results being obtained for the real data in [9]. Section 5 concludes the paper.

Definition 1.1 (Two Person Zero Sum Game [19])**.** The strategic form, or normal form, of a two person zero sum game is denoted by a triplet (*X, Y, A*) where

1. *X* is a nonempty set, the set of strategies of Player I