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The qc Yamabe problem on non-spherical quaternionic contact manifolds

Le problème de Yamabe sur des variétés avec des structures de contact quaternioniennes non sphériques

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ABSTRACT

It is shown that the qc Yamabe problem has a solution on any compact qc manifold which is non-locally qc equivalent to the standard 3-Sasakian sphere. Namely, it is proved that on a compact non-locally spherical qc manifold there exists a qc conformal qc structure with constant qc scalar curvature.

RÉSUMÉ

Il est démontré que le problème de Yamabe dans la géométrie de contact quaternionienne a une solution sur toute variété compacte avec une structure de contact quaternionienne qui n'est pas localement équivalente à la sphère standard 3-Sasakienne. Justement, il est démontré qu'il existe une structure de contact quaternionienne avec courbure scalaire constante sur une variété compacte avec une structure de contact quaternionienne qui n'est pas localement sphérique.

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1. Introduction

It is well known that the sphere at infinity of a non-compact symmetric space M of rank one carries a natural Carnot–Carathéodory structure, see [26,27]. In the real hyperbolic case one obtains the conformal class of the round metric on the sphere. In the remaining cases, each of the complex, quaternionic and octonionic hyperbolic metrics on the unit ball induces a Carnot–Carathéodory structure on the unit sphere. This defines a conformal structure on a sub-bundle of the tangent bundle of co-dimension $\dim_{\mathbb{R}} \mathbb{K} - 1$, where $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. In the complex case the obtained geometry is the well studied standard CR structure on the unit sphere in complex space. In the quaternionic case one arrives at the notion of a quaternionic contact structure. The quaternionic contact (qc) structures were introduced by O. Biquard, see [4,5], and are modeled on the conformal boundary at infinity of the quaternionic hyperbolic space. Biquard showed that the infinite dimensional family [24] of complete quaternionic-Kähler deformations of the quaternionic hyperbolic metric have conformal infinities which provide an infinite dimensional family of examples of qc structures. Conversely, according to [4] every real analytic qc structure on a manifold M of dimension at least eleven is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of M. Furthermore, [4] considered CR and qc structures as boundaries at infinity of Einstein metrics rather than only as boundaries at infinity of Kähler–Einstein and quaternionic-Kähler metrics, respectively. In fact, in [4] it was shown that in each of the three cases (complex, quaternionic, octoninoic) any small perturbation of the standard Carnot–Carathéodory structure on the boundary is the conformal infinity of an essentially unique Einstein metric on the unit ball, which is asymptotically symmetric. In the Riemannian case the corresponding question was posed in [8] and the perturbation result was proven in [14].

There is a deep analogy between the geometry of qc manifolds and the geometry of strictly pseudoconvex CR manifolds as weel as the geometry of conformal Riemannian manifolds. The qc structures, appearing as the boundaries at infinity of asymptotically hyperbolic quaternionic manifolds, generalize to the quaternion algebra the sequence of families of geometric structures that are the boundaries at infinity of real and complex asymptotically hyperbolic spaces. In the real case, these manifolds are simply conformal manifolds and in the complex case, the boundary structure is that of a CR manifold.

A natural extension of the Riemannian and the CR Yamabe problems is the quaternionic contact (qc) Yamabe problem, a particular case of which [13,30,15,16] amounts to finding the best constant in the L^2 Folland–Stein Sobolev-type embedding and the functions for which the equality is achieved, [9] and [10], with a complete solution on the quaternionic Heisenberg groups given in [16–18].

Following Biquard, a quaternionic contact structure (*qc structure*) on a real $(4n+3)$-dimensional manifold M is a codimension three distribution H (*the horizontal distribution*) locally given as the kernel of a $\mathbb{R}^3$-valued one-form $\eta = (\eta_1, \eta_2, \eta_3)$, such that, the three two-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic Hermitian structure on H. The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$, and therefore H is equipped with a conformal class $[g]$ of quaternionic Hermitian metrics.

For a qc manifold of crucial importance is the existence of a distinguished linear connection ∇ preserving the qc structure, defined by O. Biquard in [4], and its scalar curvature S, called qc-scalar curvature. The Biquard connection plays a role similar to the Tanaka–Webster connection [29,31] in the CR case. A natural question coming from the conformal freedom of the qc structures is the *quaternionic contact Yamabe problem [30,15,20]: The qc Yamabe problem on a compact qc manifold M is the problem of finding a metric $\bar{g} \in [g]$ on H for which the qc-scalar curvature is constant.*

The question reduces to the solvability of the quaternionic contact (qc) Yamabe equation

$$\mathcal{L} f := 4\frac{Q+2}{Q-2} \triangle f - f\, S \;=\; -\, f^{2^*-1}\, \overline{S},$$

where $\triangle$ is the sub-Laplacian, $\triangle f \;=\; tr^g(\nabla^2 f)$, S and $\overline{S}$ are the qc-scalar curvatures correspondingly of (M, η) and $(M, \bar{\eta} = f^{4/(Q-2)}\eta)$, where $2^* = \frac{2Q}{Q-2}$, with $Q = 4n + 6$-the homogeneous dimension.