J. Math. Pures Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Existence and nonexistence of solutions for the heat equation with a superlinear source term

Yohei Fujishima [a], Norisuke Ioku [b,*]

[a] Department of Mathematical and Systems Engineering, Faculty of Engineering, Shizuoka University, 3-5-1 Johoku, Hamamatsu 432-8561, Japan
[b] Graduate School of Science and Engineering, Ehime University, Matsuyama, Ehime 790-8577, Japan

ARTICLE INFO

Article history:
Received 18 January 2017
Available online xxxx

MSC:
primary 35K55
secondary 35A01, 46E30

Keywords:
Nonlinear heat equation
Scale invariance
Singular initial data
Existence and nonexistence

ABSTRACT

We develop a classification theory for the existence and non-existence of local in time solutions for initial value problems for nonlinear heat equations. By focusing on some quasi-scaling property and its invariant integral, we reveal the explicit threshold integrability of initial data that classifies the existence and nonexistence of solutions. Typical nonlinear terms, for instance polynomial type, exponential type and their sums, products and compositions can be treated as applications.

RÉSUMÉ

Nous développons une théorie de classification pour l'existence ou la non-existence de solutions locales en temps pour des problèmes de valeur initiale pour des équations de la chaleur non linéaires. En utilisant une propriété de quasi-scaling et son intégrale invariante, nous explicitons le seuil d'integrabilité des données initiales qui classifie l'existence et la non-existence de solutions. Les termes non linéaires typiques, par exemple du type polynomial, exponentiel et leurs sommes, produits et compositions peuvent être traités comme des applications.

1. Introduction

We consider the existence and nonexistence of solutions for the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: fujishima@shizuoka.ac.jp (Y. Fujishima), ioku@ehime-u.ac.jp (N. Ioku).

https://doi.org/10.1016/j.matpur.2018.08.001
0021-7824/© 2018 Published by Elsevier Masson SAS.

where $\partial_t = \partial/\partial t$, $N \geq 1$, $T > 0$, u_0 is a nonnegative measurable initial function and $f \in C^1([0,\infty))$ is a positive monotonically increasing function in $(0,\infty)$, that is,

$$f(s) > 0, \quad f'(s) \geq 0 \quad \text{for all } s \in (0,\infty). \tag{1.2}$$

In what follows, for a suitable Banach space X, we say that a function $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ is a classical solution in X for the problem (1.1) if u satisfies the equation in the classical sense and $\|u(t) - e^{t\Delta}u_0\|_X \to 0$ as $t \to 0$, where $e^{t\Delta}u_0$ denotes the solution to the linear heat equation with the initial data u_0.

It is well known that the problem (1.1) possesses a unique classical solution in $L^\infty(\mathbb{R}^N)$ for general nonlinearities $f \in C^1([0,\infty))$ if $u_0 \in L^\infty(\mathbb{R}^N)$. On the other hand, for the case $u_0 \notin L^\infty(\mathbb{R}^N)$, the existence of solutions heavily depends on the growth rate of the nonlinear term f. One of the typical examples of f is a power type nonlinearity, that is,

$$\begin{cases} \partial_t u = \Delta u + u^p, & x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $p > 1$. For unbounded initial data, this equation has been studied intensively since the pioneering work of Weissler [38]. In particular it is well know that:

- if $r = \frac{N}{2}(p-1) > 1$ or $r \geq 1$ satisfies $r > \frac{N}{2}(p-1)$, then for any $u_0 \in L^r(\mathbb{R}^N)$, there exist a constant $T > 0$ and a local in time classical solution $u \in C([0,T); L^r(\mathbb{R}^N))$ for the problem (1.3).
- if $1 \leq r < \frac{N}{2}(p-1)$, then there exists an initial function $u_0 \in L^r(\mathbb{R}^N)$ such that the problem (1.3) cannot possess any local in time nonnegative classical solutions.

See [1], [3], [4], [11–14], [16–19], [24], [28], [30–38] for the existence and nonexistence of solutions for nonlinear parabolic equations and their qualitative properties. We also refer to [21] and [29], which include further numerous references on the topic.

One of the most important properties of the problem (1.3) is an invariance property of the equation under the scaling

$$u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \qquad \lambda > 0. \tag{1.4}$$

If u satisfies $\partial_t u = \Delta u + u^p$, then so does u_λ. Then we can find the scale invariant norm under the scaling (1.4). In fact, $\|u_\lambda(\cdot,0)\|_{L^r(\mathbb{R}^N)} = \|u(\cdot,0)\|_{L^r(\mathbb{R}^N)}$ if and only if $r = r_c$. Therefore, the critical exponent $r_c := \frac{N}{2}(p-1)$, which gives the classification of the existence and nonexistence of solutions for the problem (1.3), arises from the scale invariance property. Although this scaling property for the equation is absent in the problem (1.1) in general, we are still able to propose an integral quantity, invariance of which under suitable transformations generalizes the scaling (1.4). In this paper, we focus on the quasi-scaling proposed by the first author in [8], and study the threshold integrability of u_0 in order to classify the existence and nonexistence of solutions for (1.1), which is determined by the behavior of $f(s)$ as $s \to \infty$. To this end, we introduce the quasi-scaling:

$$u_\lambda(x,t) := F^{-1}\left[\lambda^{-2} F(u(\lambda x, \lambda^2 t))\right] \quad (\lambda > 0), \quad \text{where } F(s) := \int_s^\infty \frac{du}{f(u)} \tag{1.5}$$

and F^{-1} is the inverse function of F. We note here that the transformation (1.5) does not preserve the equation (1.1). In fact, for a solution u of (1.1), the function u_λ defined by (1.5) satisfies

Please cite this article in press as: Y. Fujishima, N. Ioku, Existence and nonexistence of solutions for the heat equation with a superlinear source term, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.08.001