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Algebraic dimension of complex nilmanifolds

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ABSTRACT

Let M be a complex nilmanifold, that is, a compact quotient of a nilpotent Lie group endowed with an invariant complex structure by a discrete lattice. A holomorphic differential on M is a closed, holomorphic 1-form. We show that $a(M) \leq k$, where $a(M)$ is the algebraic dimension $a(M)$ (i.e. the transcendence degree of the field of meromorphic functions) and k is the dimension of the space of holomorphic differentials. We prove a similar result about meromorphic maps to Kähler manifolds.

1. Introduction

1.1. Nilmanifolds: definition and basic properties

A **nilmanifold** is a compact manifold equipped with a transitive action of a nilpotent Lie group. As shown by Malcev ([20]), every nilmanifold can be obtained as a quotient of a nilpotent Lie group G by a discrete lattice Γ. Moreover, the group G can be obtained as so-called Malcev completion of Γ, that is, as a product of exponents of formal logarithms of elements Γ. Therefore, any nilmanifold is uniquely determined by its fundamental group, which is a discrete nilpotent torsion-free group, and any such group uniquely determines a nilmanifold.

Since the work of Thurston ([29]), geometric structures on nilmanifolds are used to provide many interesting examples (and counterexamples) in complex and symplectic geometry [1,8,10,15,17]. It was Thurston

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who realized that the Kodaira surface (also known as a Kodaira–Thurston surface) is symplectic, but does not admit any Kähler structure. In this way Thurston obtained a counterexample to a result stated by H. Guggenheimer ([16]) in 1951. Guggenheimer claimed that the Hodge decomposition is true for compact symplectic manifolds, but for symplectic nilmanifolds this is usually false.

Before 1990-ies, a "complex nilmanifold" meant a compact quotient of a complex nilpotent Lie group by a discrete, co-compact subgroup. The first non-trivial example is given by so-called Iwasawa manifold ([14]) which is obtained as a quotient of the 3-dimensional Lie group of upper triangular 3 by 3 matrices by a discrete co-compact subgroup, for example the group of upper triangular matrices with coefficients in $\mathbb{Z}[\sqrt{-1}\,]$.

Starting from late 1990-ies, a "complex nilmanifold" means a quotient of a real nilpotent Lie group equipped with a left-invariant complex structure by the left action of a discrete, co-compact subgroup ([9]). This is the notion we are going to use in this paper. This definition is much more general, indeed, left-invariant complex structures are found on many even-dimensional nilpotent Lie groups which are not complex. The complex structure on a Kodaira surface is one of such examples.

Complex structures on a nilmanifold have a very neat algebraic characterization. Let G be a real nilpotent Lie group, and $\mathfrak{g}$ is Lie algebra. By Newlander–Nirenberg theorem, a complex structure on G is the same as a sub-bundle $T^{1,0}G \subset TG \otimes_{\mathbb{R}} \mathbb{C}$ such that $[T^{1,0}G, T^{1,0}G] \subset T^{1,0}G$ and $T^{1,0}G \oplus \overline{T^{1,0}G} = TG \otimes_{\mathbb{R}} \mathbb{C}$. The left-invariant sub-bundles in $T^{1,0}G$ are the same as subspaces $W \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and the condition $[T^{1,0}G, T^{1,0}G] \subset T^{1,0}G$ is equivalent to $[W, W] \subset W$. Therefore, left-invariant complex structures on G are the same as complex sub-algebras $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

A real nilmanifold is obtained as an iterated fibration with fibers which are compact tori. It is natural to ask if any complex nilmanifold can be obtained as an iterated fibration with fibers which are complex tori. The answer is negative: see e.g. [25].

However, a weaker statement is still true. If we replace fibrations of nilmanifolds by homomorphisms of their Lie algebras, it is possible to construct a homomorphism $\psi: \mathfrak{g} \longrightarrow \mathfrak{a}$ to a complex abelian Lie algebra compatible with a complex structure. Since $\mathfrak{a}$ is abelian, $\ker\psi$ necessarily contains the commutator $[\mathfrak{g}, \mathfrak{g}]$. Since it is complex, $\ker\psi$ contains $[\mathfrak{g}, \mathfrak{g}] + I[\mathfrak{g}, \mathfrak{g}]$.

The quotient algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] + I[\mathfrak{g}, \mathfrak{g}]$ is called **the algebra of holomorphic differentials on** G, denoted by $\mathfrak{H}^1(M)$. Its dimension is always positive ([27]).

1.2. Algebraic dimension and holomorphic differentials

Recall that a positive closed (1,1)-current T on a complex manifold is said to have **analytic singularities** (see [5]) if locally $T = \theta + dd^c\varphi$ for a smooth form θ and a plurisubharmonic function $\varphi = c\log(|f_1|^2 + ... + |f_n|^2)$ where $f_1, ... f_n$ are analytic functions and c a constant. Such currents have decomposition into absolutely continuous and singular part, where the absolutely continuous part is positive and closed.

Definition 1. Let M be a complex manifold. The **Kähler rank** $k(M)$ of M is the maximal rank of the absolutely continuous part of a positive, closed (1,1)-current on M with analytic singularities.

Definition 2. The **algebraic dimension** $a(M)$ of a complex manifold is the transcendence degree of its field of meromorphic functions.

Let X be a complex manifold, and $\varphi: X \dashrightarrow \mathbb{C}^n$ a meromorphic map defined by generators of the field of meromorphic functions. An algebraic reduction of X is a compactification of $\varphi(X)$ in $\mathbb{C}P^n \supset \mathbb{C}^n$. It is known to be a compact, algebraic variety ([6,30]).

We should note that the map φ is defined for more general spaces X. For smooth manifolds we'll use the following [23, Definition–Theorem 6.5].

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