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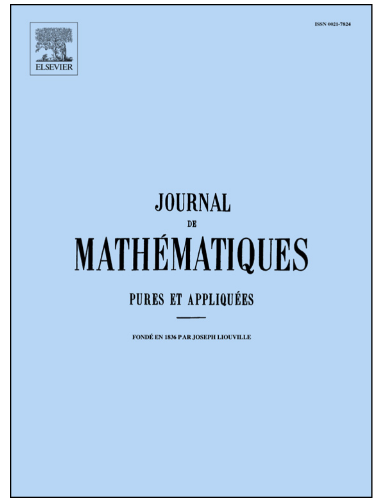

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Samer Dweik

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LACK OF REGULARITY OF THE TRANSPORT DENSITY IN THE MONGE PROBLEM

SAMER DWEIK

ABSTRACT. In this paper, we provide a family of counter-examples to the regularity of the transport density in the classical Monge-Kantorovich problem. We prove that the $W^{1,p}$ regularity of the source and target measures $f^\pm$ does not imply that the transport density σ is $W^{1,p}$, that the BV regularity of $f^\pm$ does not imply that σ is BV and that $f^\pm \in C^\infty$ does not imply that σ is $W^{1,p}$, for large p.

1. INTRODUCTION

The mass transport problem dates back to a work from 1781 by Gaspard Monge, *Mémoire sur la théorie des déblais et des remblais* ([16]), where he formulated a natural question in economics which deals with the optimal way of moving points from one mass distribution to another so that the total work done is minimized. In his work, the cost of moving one unit of mass from x to y is measured with the Euclidean distance $|x-y|$, even though many other cost functions have been studied later on.

In order to explain this problem in full details, let us consider $f^\pm$ two given finite positive Borel measures in Ω satisfying the mass balance condition $f^+(\Omega) = f^-(\Omega)$, where Ω is a compact convex set in $\mathbb{R}^d$. Let $|.|$ stand for the Euclidean norm in $\mathbb{R}^d$. Then, the classical Monge optimal transportation problem ([16]) consists in finding a transport map $T : \Omega \mapsto \Omega$ minimizing the functional

$$T \mapsto \int_\Omega |x - T(x)| \,\mathrm{d}f^+,$$

among all Borel measurable maps $T : \Omega \mapsto \Omega$ which satisfy the "push-forward" condition $T_\# f^+ = f^-$, i.e

$$f^-(A) = f^+(T^{-1}(A)) \text{ for every Borel set } A \subset \Omega.$$

The existence of optimal maps was addressed by many authors [1], [4], [10], [17] and [20] (see [5] for the most general result, which is valid for arbitrary norms $||x-y||$; however in this paper we will concentrate only on the Euclidean case). Although this problem may have no solutions, its relaxed setting (which is the Kantorovich problem [15]) always has one. The relaxed problem consists in finding a Borel measure λ over $\Omega \times \Omega$ (called *optimal transport plan*) satisfying $\pi^\pm_\# \lambda = f^\pm$, where $\pi^\pm : \Omega \times \Omega \mapsto \Omega$ being the projections on the first and second factor, respectively (i.e $\pi^\pm(x^+, x^-) := x^\pm$), which minimizes the functional

$$\lambda \mapsto \int_{\Omega\times\Omega} |x - y| \,\mathrm{d}\lambda$$

among all Borel measures λ on $\Omega \times \Omega$ satisfying $\pi^\pm_\# \lambda = f^\pm$. For the details about Optimal Transport theory, its history, and the main results, we refer to [19] and [21]. It is also possible to prove that the maximization of the functional

$$u \mapsto \int_\Omega u \,\mathrm{d}(f^+ - f^-),$$

among all the 1-Lipschitz functions u on Ω, is the dual to the Kantorovich problem (a maximizer for this problem is called *Kantorovich potential*). This duality implies that optimal λ and u satisfy