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Convolutions for localization operators

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ABSTRACT

Quantum harmonic analysis on phase space is shown to be linked with localization operators. The convolution between operators and the convolution between a function and an operator provide a conceptual framework for the theory of localization operators which is complemented by an appropriate Fourier transform, the Fourier–Wigner transform. We link the Hausdorff–Young inequality for the Fourier–Wigner transform with Lieb's inequality for ambiguity functions. Noncommutative Tauberian theorems due to Werner allow us to extend results of Bayer and Gröchenig on localization operators. Furthermore we show that the Arveson spectrum and the theory of Banach modules provide the abstract setting of quantum harmonic analysis.

RÉSUMÉ

On montre que l'analyse harmonique sur l'espace des phases est liée aux opérateurs de localisation. La convolution entre opérateurs d'une part, et entre fonction et opérateur d'autre part fournit un cadre conceptuel pour la théorie des opérateurs de localisation. Cette théorie est complétée par l'usage d'une transformée de Fourier adéquate : la transformée de Fourier–Wigner. Un lien est établi entre les inégalités de Hausdorff–Young portant sur la transformée de Fourier–Wigner, et les inégalités de Lieb relatives aux fonctions d'ambigüité. Des théorèmes taubériens non-commutatifs dus à Werner permettent d'étendre des résultats de Bayer et Gröchenig sur les opérateurs de localisation. De plus, on montre que le spectre d'Arveson et la théorie des modules de Banach fournit la base théorique pour l'analyse harmonique quantique.

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1. Introduction

Localization operators are operators of the form

$$\mathcal{A}_f^{\varphi_1,\varphi_2}\phi = \int_{\mathbb{R}^{2n}} f(z)\langle \phi, \pi(z)\varphi_1\rangle \pi(z)\varphi_2 \, dz,$$

for some window functions φ_1, φ_2, $\pi(z)\phi(t) = e^{2\pi i\omega t}\phi(t-x)$ and a mask f. Boundedness properties on modulation spaces and other function spaces have received some attention during the last years, see for example [3,7,10,11].

We contribute a conceptual approach to these localization operators based on Werner's theory of quantum harmonic analysis on phase space [34]. Our presentation of [34] is based on terminology and notation from time-frequency analysis and harmonic analysis. Various proofs in [34] are just indicated and we have decided to include complete proofs for all the major results in Werner's theory. We hope that in this way the deep results by Werner become more accessible to a wider audience.

Convolutions of functions with operators and of operators with operators have been introduced in [34], along with a corresponding Fourier transform of operators – the Fourier–Wigner transform. The convolution is based on a natural notion of translation of an operator via conjugation of an operator by the Schrödinger representation. Localization operators and the Berezin transform are expressed in terms of convolutions between a function and an operator, and between two operators. The Fourier–Wigner transform behaves like the one for functions as it maps convolutions into products, there is a Riemann–Lebesgue lemma and a Hausdorff–Young inequality [34]. We complement Werner's results with the observation that the Hausdorff–Young inequality in the rank-one case yields Lieb's uncertainty principle with constant one [30]. Hence if one invokes Lieb's inequality in the proof of the Hausdorff–Young inequality for the Fourier–Wigner transform and the singular value decomposition, then one obtains a sharp inequality for trace class operators in certain cases.

Using these concepts we formulate and prove a version of Wiener's Tauberian theorem for operators due to Werner. These variants of Tauberian theorems have shown to be of relevance in quantum mechanics and quantum information theory [25,26].

The main novel contribution is a formulation of localization operators using the convolution of a function with a rank-one operator, which gives an extension of results in [3] on the following problem: what conditions must be imposed on the windows φ_1, φ_2 to guarantee that the set $\{A_f^{\varphi_1,\varphi_2} \,|\, f \in L^1(\mathbb{R}^{2d})\}$ is dense in different spaces of operators?

In addition, the theory of Banach modules is used to prove new results on the convolutions, and the Fourier–Wigner transform is shown to be related to the Arveson spectrum. Finally the convolutions are considered in the context of modulation spaces, inspired by the existing literature on localization operators and modulation spaces.

2. Prerequisites

2.1. Notation and conventions

If X is a Banach space we will denote its dual space by X^*, and for $x \in X$ and $x^* \in X^*$ we write $\langle x^*, x\rangle$ to denote $x^*(x)$. In order to agree with inner product notation, the duality bracket $\langle \cdot, \cdot \rangle$ will always be antilinear in the second argument.

Elements of $\mathbb{R}^{2d}$ will often be written in the form $z = (x, \omega)$ for $x, \omega \in \mathbb{R}^d$. Functions on $\mathbb{R}^d$ and $\mathbb{R}^{2d}$ will often play different roles, so we denote functions on $\mathbb{R}^d$ by Greek letters such as ψ, ϕ, and functions on $\mathbb{R}^{2d}$ by Latin letters such as f, g. If ψ and ϕ are functions on $\mathbb{R}^d$, then we write $\psi \otimes \phi$ for the function on $\mathbb{R}^{2d}$