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Encoding and avoiding 2-connected patterns in polygon dissections and outerplanar graphs

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ABSTRACT

Let $\varDelta = \{\delta_1, \delta_2, \ldots, \delta_m\}$ be a finite set of 2-connected patterns, i.e. graphs up to vertex relabelling. We study the generating function $D_\varDelta(z, u_1, u_2, \ldots, u_m)$, which counts polygon dissections and marks subgraph copies of δ_i with the variable u_i. We prove that this is always algebraic, through an explicit combinatorial decomposition depending on $\varDelta$. The decomposition also gives a defining system for $D_\varDelta(z, \mathbf{0})$, which encodes polygon dissections that avoid these patterns as subgraphs. In this way, we are able to extract normal limit laws for the patterns when they are encoded, and perform asymptotic enumeration of the resulting classes when they are avoided. The results can be transferred to the case of labelled outerplanar graphs. We give examples and compute the relevant constants when the patterns are small cycles or dissections.

1. Introduction

The study of subgraph appearances in random graph models is a well established line of research, beginning with the classic *Erdős–Rényi* graph and results concerning the distribution of such appearances and threshold phenomena, as in [15,21]. In parallel, attention was also drawn on models where, given some well-known graph class, an object is chosen uniformly at random from all the objects of size n; see for instance [16] and [11] for regular graphs. In the last decades, techniques using a mixture of generating function theory and analytic tools have evolved significantly and are in the centre of such advances for various other graph classes. A number of graph statistics, such as number of components, edges, cut vertices, triangles, chromatic number and others, have been studied for standard graph classes, such as planar graphs, outerplanar, series–parallel, graphs of fixed genus, and minor-closed families; see for instance [4,2,13,17].

In [8], the authors present a normality result for the so-called *subcritical* family of graphs, that contains standard graph classes such as trees, cacti graphs, outerplanar, and series–parallel graphs. In particular, all subgraph parameters in such a class follow a normal limit law, with linear mean and variance. However, no constructive way is given in it, in order to compute the corresponding constants for the mean and variance. One of the results of this work is an explicit way to do so in outerplanar graphs, for any set of 2-connected patterns, i.e. graphs up to vertex relabelling. As a case study, we examine 3 and 4-cycles, but the process by which these constants are obtained can be directly transferred to the case of any set of 2-connected parameters.

Theorem 1.1. *The number of appearances X_n of 3-cycles and 4-cycles in polygon dissections and outerplanar graphs of size n follows a normal limit law, as in 2.2, where the mean and variance are asymptotically linear, i.e. $\mathbb{E}[X_n] = \mu n + \mathcal{O}(1)$ and*

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$\mathbb{V}ar[X_n] = \sigma^2 n + \mathcal{O}(1)$. *The constants* μ *and* σ^2 *are the following, in their exact values for dissections and in approximation for outerplanar graphs:*

Parameter	μ	σ^2	μ	σ^2
3-cycles	$\frac{1}{2}$	$\frac{-13+9\sqrt{2}}{-12+8\sqrt{2}} \approx 0.39644$	0.34793	0.40737
4-cycles	$\frac{-30+21\sqrt{2}}{-12+8\sqrt{2}} \approx 0.43933$	$\frac{-24216+17123\sqrt{2}}{-32\left(-3+2\sqrt{2}\right)^2} \approx 0.44710$	0.33705	0.36145

A necessary step for the analysis of outerplanar graphs is the analysis of polygon dissections, denoted by $\mathcal{D}$, with some fixed numbering on the vertices. For a finite set of 2-connected patterns $\Delta = \{\delta_1, \delta_2, \ldots, \delta_k\}$, we prove a combinatorial decomposition of $\mathcal{D}$ that allows the encoding of such patterns, depending on Δ. In this way, we obtain defining systems for the multivariate generating function $D_\Delta(z, u_1, u_2, \ldots, u_k)$, where the coefficient of $z^n u_1^{n_1} \cdots u_m^{n_m}$ counts the number of $\alpha \in \mathcal{D}$ that have n vertices and n_i subgraph occurrences of the pattern δ_i.

This task is of independent interest, as it is related to the enumeration problem of polygon dissections, a line of work that is quite old. Starting from the enumeration of polygon triangulations with Euler and Segner in the 18th century, a great amount of work has been devoted up until today to relevant problems. Usually, these problems put restrictions either on the number or the size of the partition's polygonal components, or even colour restrictions, recently; see for instance [3,20,1]. However, the problem where a whole pattern is avoided as subgraph (i.e., cannot be recovered by applying edge and vertex deletions) seems to not have been studied at all, except for the case of triangle freeness in [1], where the problem the authors are dealing with does not concern subgraph restrictions, but restrictions on the type and colour of the partition's polygonal components. With results of this work, it is possible to handle subgraph restrictions of any set Δ and perform asymptotic enumeration of the resulting classes. We give such examples. In fact, we obtain the following results (corresponding to Corollary 3.3.1 and Theorem 4.2, respectively):

Theorem 1.2. *The generating function* $D_\Delta(z, \mathbf{u})$ *is algebraic and the defining polynomial is computable. The generating function of polygon dissections that avoid all* Δ*-patterns as subgraphs,* $D_\Delta(z, \mathbf{0})$*, is likewise algebraic.*

Theorem 1.3. *Let* $\mathcal{D}, \mathcal{G}$ *be the classes of dissections and outerplanar graphs avoiding a set of 2-connected patterns* $\Delta = \{\delta_1, \ldots, \delta_m\}$*, respectively. Then,* $\mathcal{D}$ *and* $\mathcal{G}$ *have asymptotic growth of the form:*

$$\alpha_n \sim \frac{\alpha}{\Gamma(-\frac{1}{2})} \cdot n^{-3/2} \cdot r^{-n} \quad \text{and} \quad g_n \sim \frac{g}{\Gamma(-\frac{3}{2})} \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!,$$

respectively, where both α, g *are computable constants. In Table 3, there are approximations of* α, g *for various choices of* Δ.

We also prove a multivariate central limit theorem for the number of appearances of 2-connected patterns in polygon dissections (corresponding to Theorem 3.4):

Theorem 1.4. *Let* $\Delta = \{\delta_1, \ldots, \delta_m\}$ *be a set of 2-connected patterns. Let* Ω_n *be the set of polygon dissections of size n and* $X_n : \Omega_n \to \mathbb{Z}_{\geq 0}^m$ *be a vector of random variables* $X_{\delta_1}, \ldots, X_{\delta_m}$ *in* Ω_n*, such that* $X_{\delta_i}(\omega)$ *is the number of* δ_i *patterns in* $\omega \in \Omega_n$*. Then,* $\mathbf{X}_n$ *satisfies a central limit theorem*

$$\frac{1}{\sqrt{n}}(\mathbf{X}_n - \mathbb{E}[\mathbf{X}_n]) \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma})$$

with

$$\mathbb{E}[\mathbf{X}_n] = \boldsymbol{\mu} n + \mathcal{O}(1) \text{ and } \mathbb{C}ov[\mathbf{X}_n] = \mathbf{\Sigma} n + \mathcal{O}(1),$$

where $\boldsymbol{\mu}$ *and* $\mathbf{\Sigma}$ *are computable.*

There are some natural questions arising from this work. One is whether it is possible to extend the combinatorial construction that is proved for general parameters, with multiple cut vertices, and how. Also, one might wonder in which other combinatorial structures we can apply this reasoning, apart from outerplanar graphs. An example for the latter can be found in the dual class of polygon dissections, planted plane trees with outdegrees in $\mathbb{N} \setminus \{1\}$, denoted by $\mathcal{T}$. Consider as parameter in $T \in \mathcal{T}$ the number of subtrees T' with k leaves, such that $\deg_T(v) = \deg_{T'}(v)$ for each node v that is inner in T'. Then, the equivalent parameter for polygon dissections is the number of k-cycles.

Plan of the paper. In Section 2, we mention definitions and theorems that will be used. In Section 3, we prove a combinatorial decomposition of $\mathcal{D}$ depending on Δ and then Theorem 1.2. We also prove Theorem 1.4. In Section 4, we give applications of the previous and prove Theorems 1.1 and 1.3. In the Appendix, Table A.1 contains the initial terms of all the counting sequences appearing in Section 4.

2. Preliminaries

The framework we use is the *symbolic method* and the corresponding analytic techniques, as they were presented in [9].