Topology and its Applications 249 (2018) 1–8

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Complete ω-balancedness in semitopological groups

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A R T I C L E I N F O

Article history:
Received 23 July 2018
Received in revised form 6 September 2018
Accepted 8 September 2018
Available online 11 September 2018

MSC:
54A25
54H11
54B10

Keywords:
Semitopological group
Paratopological group
Strongly metrizable
Completely ω-balanced
ω-Balanced

A B S T R A C T

We introduce the notion of completely ω-balanced semitopological group and prove that this property is preserved under subgroups and topological products. We also show that a regular semitopological group G admits a homeomorphic embedding as a subgroup into a product of strongly metrizable semitopological groups if and only if G is completely ω-balanced and $Ir(G) \leq \omega$.

1. Introduction

The class of strongly metrizable spaces is closed under subspaces and countable products. Besides, several properties of separable metrizable spaces can be extended to the class of strongly metrizable spaces [10,11]. In [3,4], the authors initiated the study of topological groups which can be embedded as subgroups into products of strongly metrizable groups. In [3] is defined the notion of completely ω-balanced topological group to prove that a topological group G can be embedded as subgroups into products of strongly metrizable groups if and only if G is completely ω-balanced (see [3, Theorem 4.1]).

The notion of complete ω-balancedness is between ω-narrowness and ω-balancedness [3, Propositions 3.5 and 3.6] in the following sense: every ω-narrow group is completely ω-balanced and each completely

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[1] The first author was supported by CONACYT (Mexico), grant number 57142.

https://doi.org/10.1016/j.topol.2018.09.005

ω-balanced group is ω-balanced, and these three concepts are different each other (see [4, Examples 4.7 and 4.8] and [3, Proposition 4.2]).

Recently, the authors of [5] give some sufficient conditions under which a paratopological group is topological isomorphic to a subgroup of a product of strongly metrizable paratopological groups.

Our purpose is to generalize the concept of complete ω-balancedness to the realm of paratopological groups and semitopological groups. We prove that the class of completely ω-balanced semitopological groups is closed under taking subgroups and products. In addition, we extend the following relation to paratopological groups (see Propositions 3.8 and 3.9):

$$\text{totally } \omega\text{-narrow} \Rightarrow \text{completely } \omega\text{-balanced} \Rightarrow \omega\text{-balanced}.$$

We prove that a regular semitopological group G admits a homeomorphic embedding as a subgroup into a product of strongly metrizable semitopological groups if and only if G is completely ω-balanced and $Ir(G) \leq \omega$ (see Theorem 3.12).

2. Preliminaries

Given a topology on a group G, then both structures must be compatible in the following sense: A (*semitopological*) *paratopological* group G is a group with a topology such that the multiplication is (separately) continuous. In addition, if in a paratopological group G the inversion function is continuous, then it is called a *topological group*.

For a paratopological group G with topology τ, the *conjugate topology* on G is $\tau^{-1} = \{U^{-1} : U \in \tau\}$. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on G and $G^* = (G, \tau^*)$ is called the topological group *associated to* G [8].

Given a paratopological group G with identity e, the symbol $\mathcal{N}(e)$ denotes the family of open neighborhoods of e. Remember that a paratopological group G is called *ω-narrow* if for each $V \in \mathcal{N}(e)$ there exists a countable subset $A \subset G$ such that $AV = VA = G$ and it is called *totally ω-narrow* if G^* is ω-narrow. A paratopological group is *ω-balanced* if for each $U \in \mathcal{N}(e)$ there exists a countable subfamily γ of $\mathcal{N}(e)$ such that for each $g \in G$ there exists $V \in \gamma$ satisfying $gVg^{-1} \subset U$.

A subset V of a paratopological group G is called *ω-good* if there exists a countable family $\gamma \subset \mathcal{N}(e)$ such that for every $x \in V$, we can find $W \in \gamma$ with $xW \subseteq V$. The ω-good subsets were defined in [8]. The following result permits us to prove Theorem 3.12.

Lemma 2.1. ([8, Lemma 3.10]) *Every paratopological group G has a local base at the neutral element consisting of ω-good sets.*

Given a paratopological group G with identity e, we denote by $\mathcal{N}^*(e)$ the family of ω-good sets containing the identity. By Lemma 2.1, the family $\mathcal{N}^*(e)$ is a local base for G at e.

A family $\mathcal{U}$ of non-empty subsets of a paratopological group G is *dominated* by a family $\gamma \subseteq \mathcal{N}(e)$ if for every $U \in \mathcal{U}$ and $x \in U$ there exists $V \in \gamma$ such that $xV \subseteq U$ [7].

A family $\mathcal{U}$ of subsets of a set X is *star-countable* if each $U \in \mathcal{U}$ intersects only countably many members of $\mathcal{U}$. If $\mathcal{U}$ can be decomposed into a countable union of star-countable covers of X, we call it a *σ-star-countable cover of X*.

A regular space X is called *strongly metrizable* if it has a σ-star-countable base. It follows that every strongly metrizable space is metrizable.

A family $\mathcal{V}$ of subsets of a set X is a *weak refinement* of a cover $\mathcal{U}$ if $\mathcal{V}$ contains a subfamily which is a cover of X and a refinement of $\mathcal{U}$. Clearly, every base for a space X is an open weak refinement of any open cover of X.