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Remainders of extremally disconnected spaces and related objects

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ABSTRACT

In this article, we study remainders of extremally disconnected spaces and extremally disconnected remainders. Extremally disconnected spaces satisfying certain homogeneity type conditions are at the center of it. It is established that the absolute X of a non-discrete separable metrizable space M not only is never homogeneous, but never has a homogeneous extension (Theorem 3.2). This extends a classic theorem of Z. Frolik on nonhomogeneity of any extremally disconnected compactum [17], [18]. It is proved that if X is an extremally disconnected space with a homogeneous extension, then, for any compactification bX of X, the remainder $Y = bX \setminus X$ is countably compact (Theorem 3.7). Among other new results are Theorem 4.6 on remainders of Banach spaces, Theorem 4.8 on remainders of normal topological groups, and Theorem 2.9 on remainders of k-trivial spaces.

1. Preliminaries and introduction

All spaces under discussion are assumed to be Tychonoff. A space is *extremally disconnected* if the closure of each open set is open. In terminology and notation we follow [16]. In particular, ω denotes the set of all non-negative integers, and $\beta\omega$ is the Čech–Stone compactification of the discrete space ω. A space X is an *extension* of a space Y if Y is a dense subspace of X. A subspace Y of a space X is C^**-embedded* in X if every bounded continuous real-valued function on Y can be continuously extended to X. A space X is called an *absolute* of a space Y if there exists a perfect irreducible mapping f of X onto Y [25], [26], [15], where "irreducible" means that, for any proper closed subset A of X, $f(A) \neq Y$.

In this article, we study remainders of extremally disconnected spaces and remainders of some related spaces. A *compactification* of a space X is any compact space bX such that X is a dense subspace of bX. A *remainder* Y of X is the subspace $Y = bX \setminus X$ of a compactification bX. A space X is *homogeneous* if for any points $x, y \in X$ there exists a homeomorphism h from X onto itself such that $h(x) = y$.

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Of course, every discrete space is extremally disconnected. It is not so easy to visualize a non-empty extremally disconnected space without isolated points, since no such space can contain a non-trivial convergent sequence. A standard example of an infinite compact extremally disconnected space is the Čech–Stone compactification $\beta\omega$ of the discrete space ω. The topological structure of $\beta\omega$ is highly complicated (see [24]). It has been shown in [19][1.H, 6.M, 14.N] that the Čech–Stone compactification of any extremally disconnected space is extremally disconnected.

On the other hand, extremally disconnected spaces are much more widely spread over the world of abstract topological spaces than one would imagine. Indeed, with every space X an absolute of it is associated, which turns out to be an extremally disconnected space (see [25], [26], and, in the compact case, [19]). Z. Frolik has discovered a deep fact: no extremally disconnected compactum is homogeneous [18] (see also comments in [8]).

E.K. van Douwen was, probably, the first to observe that Frolik's methods can be used to obtain the following much more general conclusion: every compact subspace of any homogeneous extremally disconnected space is finite [12][Theorem 2(c) and the proof of it]. A special case of this theorem was independently obtained in 1967 by a different method by Arhangel'skii: every compact subspace of any extremally disconnected topological group is finite [1]. In this article, it is proved that the absolute X of any non-discrete separable metrizable space M cannot be densely embedded in a homogeneous space (see Theorem 3.2). This extends the theorem of Z. Frolik on non-homogeneity of any infinite extremally disconnected compactum.

The theorem of Comfort and van Mill [8][Theorem 0.7], saying that every pseudocompact extremally disconnected topological group is discrete, is generalized below as follows: if an extremally disconnected space X is a dense subspace of a homogeneous k-space Z, then X and Z are discrete (see Corollary 3.5).

For convenience, we call a space X *k-trivial* if every compact subspace of X is finite. It is shown that if X is a k-trivial space with no isolated points, then every remainder Y of X in a compactification bX is pseudocompact (Theorem 2.9). It is also proved that if X is an extremally disconnected space with a homogeneous extension, then, for any compactification bX, the remainder $Y = bX \setminus X$ is countably compact (Theorem 3.7).

Below we often make the general assumption that the spaces considered are nowhere locally compact. This assumption plays an essential role. Because of it, the space and the remainder interact deeper, and the space can be interpreted as the remainder of its remainder. In particular, we can reformulate one of the questions considered below as follows: which nowhere locally compact spaces have at least one extremally disconnected remainder?

2. Remainders of extremally disconnected spaces and some properties of compactness type

If every nowhere dense compact subspace of X is finite, then we say that X is *weakly k-trivial*. Obviously, the next statement holds.

Proposition 2.1. *A nowhere locally compact space is weakly k-trivial if and only if it is k-trivial.*

One of the results in this section which we often use below is the next theorem:

Theorem 2.2. *Suppose that X is a nowhere locally compact extremally disconnected space with a remainder Y in a compactification bX. Then either Y is countably compact, or X contains a topological copy of $\beta\omega$, and hence, $|X| \geq 2^c$.*

A useful step in the proof of this theorem is the next statement which seems to be a part of the folklore and has its roots in [19]: