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On the axiomatic systems of Steenrod homology theory of compact spaces

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ABSTRACT

On the category of compact metric spaces an exact homology theory was defined and its relation to the Vietoris homology theory was studied by N. Steenrod [11]. In particular, the homomorphism from the Steenrod homology groups to the Vietoris homology groups was defined and it was shown that the kernel of the given homomorphism are homological groups, which was called weak homology groups [11], [3]. The Steenrod homology theory on the category of compact metric pairs was axiomatically described by J. Milnor. In [10] the uniqueness theorem is proved using the Eilenberg–Steenrod axioms and as well as relative homeomorphism and clusters axioms. J. Milnor constructed the homology theory on the category Top^2_C of compact Hausdorff pairs and proved that on the given category it satisfies nine axioms — the Eilenberg–Steenrod, relative homeomorphism and cluster axioms (see theorem 5 in [10]). Besides, using the construction of weak homology theory, J. Milnor proved that constructed homology theory satisfies partial continuity property on the subcategory Top^2_{CM} (see theorem 4 in [10]) and the universal coefficient formula on the category Top^2_C (see Lemma 5 in [10]). On the category of compact Hausdorff pairs, different axiomatic systems were proposed by N. Berikashvili [1], [2], H. Inasaridze and L. Mdzinarishvili [7], L. Mdzinarishvili [9] and H. Inasaridze [6], but there was not studied any connection between them. The paper studies this very problem. In particular, in the paper it is proved that any homology theory in Inasaridze sense is the homology theory in the Berikashvili sense, which itself is the homology theory in the Mdzinarishvili sense. On the other hand, it is shown that if a homology theory in the Mdzinarishvili sense is exact functor of the second argument, then it is the homology in the Inasaridze sense.

1. Introduction

Let Top^2_C be the category of compact Hausdorff pairs and continuous maps and $\mathcal{A}b$ be the category of abelian groups.

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A sequence $\bar{H}_* = \{\bar{H}_n\}_{n\in Z}$ of covariant functors $\bar{H}_n : Top^2_C \to Ab$ is called homological [9], [4], if:

1_H) for each object $(X,A) \in Top^2_C$ and $n \in Z$ there exists a ∂-homomorphism

$$\partial : \bar{H}_n(X,A) \to \bar{H}_{n-1}(A) \tag{1.1}$$

($\bar{H}_n(A) \equiv \bar{H}_{n-1}(A,\emptyset)$, where $\emptyset$ is the empty set);

2_H) the diagram

$$\begin{array}{ccc} \bar{H}_n(X,A) & \to & \bar{H}_{n-1}(A;G) \\ \downarrow f_* & & \downarrow (f_{|A})_* \\ \bar{H}_n(Y,B) & \to & \bar{H}_{n-1}(B;G) \end{array} \tag{1.2}$$

is commutative for each continuous mapping $f : (X,A) \to (Y,B)$ ($f_* : \bar{H}_n(X,A) \to \bar{H}_n(Y,B)$ and $(f_{|A})_* : \bar{H}_n(A) \to \bar{H}_n(B)$ are the homomorphisms induced by $f : (X,A) \to (Y,B)$ and $f_{|A} : A \to B$, correspondingly).

Let $\mathcal{P}ol^2$ be the full subcategory of the category Top^2_C, consisting of compact polyhedral pairs.

A homological sequence $\bar{H}_* = \{\bar{H}_n\}_{n\in Z}$ defined on the category Top^2_C is called homology theory in the Eilenberg–Steenrod sense if it satisfies homotopy, excision, exactness and dimension axioms [4]. It is known that up to an isomorphism such a homology theory is unique on the subcategory $\mathcal{P}ol^2$ of compact polyhedral pairs [4] and it is denoted by $H_* = \{H_n\}_{n\in Z}$, but it is not unique on the category Top^2_C of compact Hausdorff pairs.

The Steenrod homology theory on the category of compact metric pairs first was axiomatically described by J. Milnor [10]. He proved the uniqueness theorem using the Eilenberg–Steenrod axioms and additionally two more — relative homeomorphism and clusters axioms:

RH (relative homeomorphism axiom): if $f : (X,A) \to (Y,B)$ is a map in Top^2_{CM} which carries $X - A$ homeomorphically onto $Y - B$, then

$$f_* : \bar{H}_n(X,A) \to \bar{H}_n(Y,B) \tag{1.3}$$

is an isomorphism.

CL(cluster axiom): if X is the union of countable many compact subsets $X_1, X_2, \dots$ which intersect pairwise at a single point $*$, and which have diameters tending to zero, then $\bar{H}_n(X,*)$ is naturally isomorphic to the direct product of the groups $\bar{H}_n(X_i,*)$.

In [10] the following is proved:

Theorem 1.1. *(see theorem 3 in [10]) Given two homology theories $\bar{H}^M_*$ and $\bar{H}_*$ on the category Top^2_{CM}, both satisfying the nine axioms (the Eilenberg–Steenrod, relative homeomorphism and cluster axioms), any coefficient isomorphism $\bar{H}^M_0(*) \approx \bar{H}_0(*) \approx G$ extends uniquely to an equivalence between the two homology theories.*

In [10] J. Milnor constructed the homology theory $\bar{H}^M_*$ on the category Top^2_C of compact Hausdorff pairs and gave its several properties. In particular the following is proved [10]:

Theorem 1.2. *(see theorem 5 in [10]) The homology theory $\bar{H}^M_*$, defined on the category Top^2_C of compact Hausdorff pairs, satisfies the nine axioms (the Eilenberg–Steenrod axioms as well as relative homeomorphism and cluster axioms).*