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The Jacobian Conjecture fails for pseudo-planes [☆]

Adrien Dubouloz [a], Karol Palka [b,*]

[a] IMB UMR5584, CNRS, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France
[b] Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-656 Warsaw, Poland

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ABSTRACT

A smooth complex variety satisfies the Generalized Jacobian Conjecture if all its étale endomorphisms are proper. We study the conjecture for $\mathbb{Q}$-acyclic surfaces of negative logarithmic Kodaira dimension. We show that G-equivariant counterexamples for infinite group G exist if and only if $G = \mathbb{C}^*$ and we classify them relating them to Belyi–Shabat polynomials. Taking universal covers we get rational simply connected $\mathbb{C}^*$-surfaces of negative logarithmic Kodaira dimension which admit non-proper $\mathbb{C}^*$-equivariant étale endomorphisms.

We prove also that for every integers $r \geq 1, k \geq 2$ the $\mathbb{Q}$-acyclic rational hyperplane $u(1+u^r v) = w^k$, which has fundamental group $\mathbb{Z}_k$ and negative logarithmic Kodaira dimension, admits families of non-proper étale endomorphisms of arbitrarily high dimension and degree, whose members remain different after dividing by the action of the automorphism group by left and right composition.

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* Corresponding author.
E-mail addresses: adrien.dubouloz@u-bourgogne.fr (A. Dubouloz), palka@impan.pl (K. Palka).

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1. Main result

The Jacobian Conjecture asserts that if an algebraic endomorphism of the complex affine space $\mathbb{A}^n = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n])$ has an invertible Jacobian then it is an isomorphism. The conjecture is open and hard, even for $n = 2$. One of the ideas which may contribute to our understanding of it is to study the problem in a broader class of varieties. A general form of the conjecture has been proposed by M. Miyanishi.

Definition 1.1 *(Generalized Jacobian Conjecture).* A smooth complex variety satisfies the *Generalized Jacobian Conjecture* if all étale endomorphisms of that variety are proper.

Unless the topological Euler characteristic of the variety is zero, properness is equivalent to the invertibility of the endomorphism (see Lemma 2.15), so for affine spaces the conjecture is equivalent to the original Jacobian Conjecture. The Generalized Jacobian Conjecture has been studied for certain classes of algebraic varieties, see Miyanishi's lecture notes [16] for a good introduction to the subject. It holds for quasi-projective varieties of log general type due to a result of Iitaka (see Lemma 2.15(4)) and it holds for curves by [14]. For surfaces there are many positive partial results, see [14], [19], [10], [20], [17], [8]. But the most intriguing question, which is in fact the main motivation, is whether the conjecture holds for varieties similar to affine spaces, similar in the sense of having the same logarithmic Kodaira dimension or the same, i.e. trivial, rational homology groups. We restrict our attention to the intersection of these two classes in dimension two, that is, to $\mathbb{Q}$-homology planes of negative logarithmic Kodaira dimension. Similarly to $\mathbb{A}^2$, they all admit an $\mathbb{A}^1$-fibration over $\mathbb{A}^1$ (hence are rational, see Proposition 2.9).