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Embeddings between lattices of fuzzy sets: An application of closed-valued fuzzy sets ☆

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Abstract

In this paper we deal with the problem of extending Zadeh's operators on fuzzy sets (FSs) to interval-valued (IVFSs), set-valued (SVFSs) and type-2 (T2FSs) fuzzy sets. Namely, it is known that seeing FSs as SVFSs, or T2FSs, whose membership degrees are singletons is not order-preserving. We then describe a family of lattice embeddings from FSs to SVFSs. Alternatively, if the former singleton viewpoint is required, we reformulate the intersection on hesitant fuzzy sets and introduce what we have called closed-valued fuzzy sets. This new type of fuzzy sets extends standard union and intersection on FSs. In addition, it allows handling together membership degrees of different nature as, for instance, closed intervals and finite sets. Finally, all these constructions are viewed as T2FSs forming a chain of lattices.

Keywords: Fuzzy sets; Interval-valued fuzzy sets; Set-valued fuzzy sets; Type-2 fuzzy sets; Closed-valued fuzzy sets

1. Introduction

The relevance of the different types of fuzzy sets, as models to handle uncertainty, is worldwide recognized. This is yet another example of the continuing need for mathematical formalisms in computer science-related fields. Coming from Zadeh's notion of Fuzzy Set (FS) [25], some of these types have been also defined in order to solve the problem of the attribution of membership degree: whenever the objects of the universe of discourse are connected to imprecise concepts, the design the membership functions might be hard to achieve. Nevertheless, as solutions to specific

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problems of different nature, their algebraic structures seem to be disconnected or, at least, their relationships are not well-understood.

In the literature there exist some attempts aiming to establish a common framework where these types of fuzzy sets could be inserted gradually into a chain of lattice structures, see the recently published survey [8] and the references therein. Nevertheless, for some cases, the compatibility between lattice structures and embedding homomorphisms is a tricky problem. The origin of our work is motivated by the problem stated in [8, Remark 2]: given a fixed universe set X, is there a lattice structure on the class of Set-Valued Fuzzy Sets (SVFSs) over X [11] extending Zadeh's union and intersection on FSs over X? Actually, this question can also be stated in the realm of Type-2 Fuzzy Sets (T2FSs) [27]. The key-point here is how we may see FSs as SVFSs, or as T2FSs. In general, FSs are viewed as SVFSs for which membership degrees are given by a singleton on $[0, 1]$. This is not consistent with the standard lattice operators, and this is why it is proposed to change the algebraic structure. Nevertheless, from an algebraic point of view, what is important are the relations (lattice morphisms) between the objects (lattices) of the category. That is, a different approach to a practical solution could then be to change the embedding and keep the lattice structures.

In this paper we study the embeddings between certain lattice structures on several types of fuzzy sets over a common universe. Concretely, we focus on FSs and the generalizations given by Interval-Valued Fuzzy Sets (IVFSs) [21], SVFSs and T2FSs. The embeddings and lattice operators under consideration are derived pointwise from their corresponding set-theoretical versions defined on the spaces where uncertainty is measured. Our main goal is to present two possible solutions to the above question. One keeps the standard lattice structures whilst the other keeps the usual embedding from FSs to SVFSs. In particular, the aims of our work are the following:

1. To show, from a mathematical perspective, why Zadeh's lattice operators on FSs and set theoretical-derived operators on SVFSs are not compatible by way of the embedding from $[0, 1]$ into its power set as singletons.
2. To provide a collection of embeddings from FSs to SVFSs making these structures compatible.
3. To reformulate the meet operator on Hesitant Fuzzy Sets (HFSs) [22] in order to obtain a lattice structure for most of their practical applications.
4. To introduce Closed-Valued Fuzzy Sets (CVFSs) in order to extend the lattice of FSs via the embedding from $[0, 1]$ into its power set as singletons.
5. To embed all these classes into T2FSs forming a chain of lattice embeddings.

The paper is organized in increasing order of generality of the different types of fuzzy sets. Namely, in Section 2 we recall the notion of FS and explain the recurrent and basic categorical constructions of the paper. Readers non-familiar with category theory may consult, for instance, the reference [16]. Section 3 concerns IVFSs. We then provide a family of embeddings from FSs to IVFSs respecting the usual meet and join operators. Sections 4 and 5 deal with the problem stated in [8, Remark 2]. In the former we describe a family of lattice embeddings from FSs to SVFSs, where SVFSs are endowed with the lattice operators derived pointwise from the set union and intersection. In the latter we show an alternative solution. We define the class of CVFSs with a lattice structure inspired by the celebrated notion of HFS. Then CVFSs become a lattice which extends the lattice of FSs through the "singleton embedding". Finally, in Section 6, we embed the achieved chains of lattices inside the class of T2FSs endowed with the structure defined in [9].

Due to the number of lattice structures we shall manage, we condense the notation used for the partial orders and the lattice operators in Table 1. There, the left column contains the operators over the classes where the uncertainty is measured, whilst the right column includes their corresponding pointwise-induced operators on the types of fuzzy sets.

2. Preliminaries

In 1965, inspired by the idea of multivalued logic, Zadeh [25] proposes the theory of Fuzzy Sets, an extension of the classical set theory, for some classes of objects whose criteria of membership are not precisely defined. In what follows, X denotes a nonempty universe set and $[0, 1]$ the closed unit interval.

Definition 1. A Fuzzy Set (FS) A on X is a set mapping $A : X \to [0, 1]$. This is also called a Type-1 Fuzzy Set (T1FS).

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