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# On efficient factorization of standard fuzzy concept lattices and attribute-oriented fuzzy concept lattices

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**Abstract**

We propose two ways to make factorization of attribute-oriented fuzzy concept lattices recently studied by Ciobanu & Văideanu more efficient. Firstly, we show that the blocks of the corresponding factor lattice are determined by a particular fuzzy interior operator. This allows us to compute an attribute-oriented fuzzy concept lattice by intents. Such a computation requires fewer traversals through data, as the number of the attributes is usually smaller than the number of objects. Secondly, we scale the input data in such a way that the factor lattice can be computed as a one-sided fuzzy-crisp concept lattice. Our experiments indicate that both methods lead to a significant improvement in efficiency of computation of the factor lattice. In addition, we improve and extend the theory provided by Ciobanu & Văideanu.


*Keywords:* Factorization; Similarity; Fuzzy sets; Formal context; Concept lattice

## 1. Introduction

Reduction of the size of concept lattice is recognized as one of the most important topics of Formal Concept Analysis (FCA). Even data which are not large can contain a large number of formal concepts. Large concept lattices are hard to read for a human user. To allow the human user to obtain useful information contained in the hierarchy of formal concepts, the number of formal concepts must be reduced. The problem is even more complex in the fuzzy setting. The reduction of fuzzy concept lattices was addressed for instance in [4,15,14,7]; see also the survey [20]. For methods of reduction of ordinary (non-fuzzy) concept lattices, see the survey [10] and references therein.

Recently, Ciobanu & Văideanu contributed to the topic by an efficient method of factorization of attribute-oriented fuzzy concept lattices. They factorized the fuzzy concept lattice by a particular tolerance relation [8]. They showed that the factors can be determined by the fixed points of a particular fuzzy closure operator and they computed them with a generalized NextClosure algorithm [2] (also used in [4]). Their results are analogous to Belohlavek et al.'s results on fast factorization of standard fuzzy concept lattices [4].

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In this paper, we improve the work of Ciobanu & Văideanu [8] in several directions. We provide short alternatives to their lengthy proofs. Specifically, we show that the results can be easily obtained from results on fast factorization of standard fuzzy concept lattices [4] using reducibility of the standard and attribute-oriented fuzzy concept lattices [5].

Furthermore, we propose two alternatives to their computation of factors. In the first one, we show that the factors are also determined by fixed points of a particular fuzzy interior operator and we alter the generalized NextClosure to compute these fixed points. This enables us to compute the factor lattice with fewer traversals through the data. In the second one, we exploit the reducibility of the standard and attribute-oriented fuzzy concept lattices and the conceptual scaling to produce data from which the factor lattice can be computed with any algorithm for enumerating fixed points of ordinary closure operators.

The paper has the following structure. We recall basic notions of fuzzy sets and formal fuzzy concept analysis in Section 2. In Section 3 we summarize results from [4] and [8]. Section 4 contains our main results: first we present simple proofs of main results from [8] in Section 4.1. Consequently, we propose the two alternatives to the computation of factors in Section 4.2 and Section 4.3. Finally, we present our conclusions in Section 5. The paper has two appendices: Appendix A contains a description of the altered NextClosure algorithm and a proof of its correctness; Appendix B contains results of our experiments.

## 2. Preliminaries

### 2.1. Complete residuated lattices

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice [3,13,21] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the partial order of $\mathbf{L}$ is denoted by $\leqslant$); $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$; $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of truth functions of “fuzzy conjunction” and “fuzzy implication.” The symbols 0 and 1 denote the least and greatest elements of $\mathbf{L}$. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Common examples of complete residuated lattices include those defined on the unit interval (i.e. $L = [0, 1]$), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$ given by $a \rightarrow b = \max\{c \mid a \otimes c \leqslant b\}$. The three most important pairs of adjoint operations on the unit interval are

- Łukasiewicz

$$a \otimes b = \max(a + b - 1, 0),$$
$$a \rightarrow b = \min(1 - a + b, 1),$$

- Gödel

$$a \otimes b = \min(a, b),$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{otherwise,} \end{cases}$$

- Goguen (product)

$$a \otimes b = a \cdot b,$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leqslant b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

Instead of a unit interval we can also consider a finite chain, e.g.

$$L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \qquad n \in \mathbb{N}^+.$$

All operations on this chain are then defined analogously, see [3].

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