Information Processing Letters 141 (2019) 9–15

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Approximability and hardness of geometric hitting set with axis-parallel rectangles

Raghunath Reddy Madireddy *, Apurva Mudgal [1]

Department of Computer Science and Engineering, Indian Institute of Technology Ropar, Rupnagar, Punjab-140001, India

ARTICLE INFO

Article history:
Received 5 November 2017
Received in revised form 8 August 2018
Accepted 10 September 2018
Available online 17 September 2018
Communicated by Ryuhei Uehara

Keywords:
Axis-parallel rectangles
Geometric hitting set
NP-hardness
APX-hardness
Computational geometry

ABSTRACT

We show that geometric hitting set with axis-parallel rectangles is APX-hard even when all rectangles share a common point. We also show that geometric hitting set problem is APX-hard for several classes of objects given in [3] such as axis-parallel ellipses with a shared point, axis-parallel cubes sharing a common point, etc.
Further, we give a polynomial time approximation scheme (PTAS) for the weighted hitting set problem with axis-parallel rectangles when all rectangles have integer side lengths bounded by a constant C. Finally, we show that the problem is NP-hard for rectangles of size 1×2 and 2×1 even when every rectangle intersects a given unit height horizontal strip.

1. Introduction

Let $\mathcal{P}$ be a set of points and $\mathcal{O}$ be a set of objects in the plane. We say that a subset $\mathcal{P}' \subseteq \mathcal{P}$ is a hitting set for $\mathcal{O}$ if every object in $\mathcal{O}$ contains at least one point in $\mathcal{P}'$. In the hitting set problem, we are given $(\mathcal{P}, \mathcal{O})$ and the goal is to find a minimum size hitting set $\mathcal{P}' \subseteq \mathcal{P}$ of points for objects in $\mathcal{O}$. In weighted version of the problem, every point $p \in \mathcal{P}$ is given a non-negative weight W_p and the goal is to find a hitting set $\mathcal{P}' \subseteq \mathcal{P}$ for $\mathcal{O}$ with minimum $\Sigma_{p \in \mathcal{P}'} W_p$. In this paper, we consider the following two hitting set problems.

Problem 1. ***Rect-Hit****. All the objects in $\mathcal{O}$ are axis-parallel rectangles.*

Problem 2. ***Rect-Hit-Bounded****. This is a special case of Rect-Hit problem where the side lengths of rectangles are integers less than or equal to C, where C is a constant.*

We now discuss some previous work and our discussion is confined to only axis-parallel rectangles.
General axis-parallel rectangles. Recently, in one of our papers [10], we show that the hitting set problem is NP-hard when $\mathcal{O}$ contains only two types of integer dimension axis-parallel rectangles and the rectangles share a common point. On the other hand, the hitting set problem is known to be APX-hard [3] even for two types of rectangles. The best-known approximation factor for hitting set with axis-parallel rectangles is $O(\log \log n)$ [2].
Bounded side length axis-parallel rectangles. The hitting set problem is known to be NP-hard for unit squares [6]. However, three PTASes are known for the hitting set problem with unit squares, one in unweighted case [12] and two by duality from PTASes for weighted set cover [5,4].

For objects in Problem 2, a PTAS is known for set cover problem [11]. However, as our objects are not unit

* Corresponding author.
E-mail addresses: raghunath.reddy@iitrpr.ac.in (R.R. Madireddy), apurva@iitrpr.ac.in (A. Mudgal).

1 Partially supported by grant No. SB/FTP/ETA-434/2012 under DST-SERB Fast Track Scheme for Young Scientist.

https://doi.org/10.1016/j.ipl.2018.09.003
0020-0190/© 2018 Elsevier B.V. All rights reserved.

squares, this does not give a PTAS for Problem 2 by duality.

The major contributions of the paper are as follows:

1. We define a special hitting set problem SPECIAL-3HS and show that it is APX hard (see Section 2).
2. We prove that Problem 1 is APX-hard even when origin in the plane is inside all rectangles by giving a reduction from SPECIAL-3HS (see Section 3). We use SPECIAL-3HS to prove APX-hardness of hitting set problem for several range spaces considered in [3].
3. We give a PTAS for weighted version of Problem 2 by using sweep-line method (see Section 4).
4. We further show that Problem 2 is NP-hard for $C = 2$ with rectangles intersecting a strip $\{(x, y) \mid 1 \leq y \leq 2\}$ (see Section 5).

2. SPECIAL-3HS is APX-hard

We start this section with the definition of the vertex cover problem. Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E is the set of edges. We say a subset $V' \subseteq V$ covers an edge $e \in E$ if at least one endpoint of e is in V'. In the vertex cover problem, the goal is to find a minimum size subset $V^* \subseteq V$ which covers all edges in E. The vertex cover problem is known to be APX-hard on cubic graphs [1].

Chan and Grant [3] define an APX-hard set cover problem SPECIAL-3SC and use it to prove APX-hardness of geometric set cover and hitting set problems with several classes of objects. In the following, we define a problem SPECIAL-3HS which can be looked as a counterpart of SPECIAL-3SC for hitting set problem. Further, we show that SPECIAL-3HS is APX-hard.

Definition 2.1. SPECIAL-3HS. Let $(\mathcal{X}, \mathcal{S})$ be a range space where $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ is a set of elements with $\mathcal{X}_1 = \{a_1, a_2, \ldots, a_n\}$, and $\mathcal{X}_2 = \{b_1, b_2, \ldots, b_{2m}\}$ such that $3n = 2m$. Further, suppose $\mathcal{S}$ is a collection of $3n + m$ subsets of elements in $\mathcal{X}$, each subset is of size two, such that for all $1 \leq p \leq m$, there exist two integers i and j ($1 \leq i < j \leq n$) such that the sets $\{a_i, b_{2p-1}\}$, $\{b_{2p-1}, b_{2p}\}$, and $\{b_{2p}, a_j\}$ are in $\mathcal{S}$. Furthermore, every element in $\mathcal{X}_1$ is exactly in three sets in $\mathcal{S}$ and every element in $\mathcal{X}_2$ is exactly in two sets in $\mathcal{S}$. The goal is to find the minimum size hitting set for $(\mathcal{X}, \mathcal{S})$.

Theorem 2.1. SPECIAL-3HS *is* APX*-hard.*

Proof. We give a reduction from the vertex cover problem on cubic graphs to SPECIAL-3HS and it consists of two parts.

First part. Let $G_1 = (V_1, E_1)$ be the given cubic graph with vertex set $V_1 = \{a_1, a_2, \ldots, a_n\}$ and edge set $E_1 = \{e_1, e_2, \ldots, e_m\}$. We now construct a graph $G_2 = (V_2, E_2)$ as follows:

1. $V_2 = A \cup B$ where $A = V_1$ and $B = \{b_1, b_2, \ldots, b_{2m}\}$.
2. For the p-th edge $e_p = \{a_i, a_j\}$ in G_1 ($1 \leq p \leq m$ and $1 \leq i < j \leq n$), we add a path of length three to graph G_2 which consists of the three edges $\{a_i, b_{2p-1}\}, \{b_{2p-1}, b_{2p}\}$, and $\{b_{2p}, a_j\}$.

One can see that G_2 is obtained by placing two new vertices on every edge in G_1. The new vertices are exactly those in set B.

From the above construction, we can note that the degree of every vertex in A is exactly three and the degree of every vertex in B is exactly two. Further, every vertex in B is adjacent to exactly one vertex in A and exactly one vertex in B. Furthermore, no two vertices in A are incident on the same edge in graph G_2.

We now show that this is a L-reduction [13] with $\alpha = 4$ and $\beta = 1$. Let $V_1^* \subseteq V_1$ and $V_2^* \subseteq A \cup B$ be the optimal vertex covers for graphs G_1 and G_2 respectively. It is a well-known observation that adding two new vertices on an edge in a graph increases the size of the optimal vertex cover by exactly one. Hence, we note that $|V_2^*| = |V_1^*| + |E_1|$.

(a) *Proof of* $\alpha = 4$. Since G_1 is a cubic graph, $|V_1^*| \geq |V_1|/2$. Therefore, $|V_2^*| \leq 4|V_1^*|$ and hence $\alpha = 4$.

(b) *Proof of* $\beta = 1$. Let $V_2' \subseteq A \cup B$ be a vertex cover of G_2. We now give a polynomial time algorithm to obtain a vertex cover V_1' of G_1 such that $||V_1'| - |V_1^*|| \leq ||V_2'| - |V_2^*||$. Initially, let $U = V_2'$.
We now do the following modifications to set U for $p = 1, 2, \ldots, m$, in this order. For the p-th edge $\{a_i, a_j\}$ in G_1 ($1 \leq i < j \leq n$), by the above reduction there exists a path $a_i - b_{2p-1} - b_{2p} - a_j$ in G_2. Since set U covers the edge $\{b_{2p-1}, b_{2p}\}$, at least one of the two vertices b_{2p-1} and b_{2p} must be in U. Now there are two cases.
1. *Exactly one among* b_{2p-1} *and* b_{2p} *is in* U. Suppose b_{2p-1} is in U. (The other case, when b_{2p} is in U, is similar.) Note that b_{2p-1} covers only $\{a_i, b_{2p-1}\}$ and $\{b_{2p-1}, b_{2p}\}$. Thus, we conclude that a_j is also in U and covers edge $\{a_i, a_j\}$ in G_1. Hence, we remove b_{2p-1} from U.
2. *Both* b_{2p-1} *and* b_{2p} *are in* U. If at least one of a_i and a_j is in U, then we remove both b_{2p-1} and b_{2p} from U. Otherwise, we remove both b_{2p-1} and b_{2p} from U and add one vertex among a_i and a_j arbitrarily to U.

Finally, set $V_1' = U$.
Clearly, V_1' is a vertex cover for G_1. For every edge in G_1, in both cases above, we are decreasing the size of U by at least one. Hence, $|V_1'| \leq |V_2'| - |E_1|$.
Therefore, $||V_1'| - |V_1^*|| \leq ||V_2'| - |E_1| - |V_1^*|| = ||V_2'| - |V_2^*||$.

Second part. We now give an encoding of G_2 into an instance of SPECIAL-3HS as follows. Let $\mathcal{X}_1 = A$ and $\mathcal{X}_2 = B$. For every edge in graph G_2, add a set of size two to $\mathcal{S}$ which consists of both endpoints of the edge. This gives a bijective mapping between vertex cover of G_2 and hitting set of $(\mathcal{X}, \mathcal{S})$.

Therefore, by combining above two parts and the fact that vertex cover on cubic graphs is APX-hard [1], we conclude that SPECIAL-3HS is APX-hard. □