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Octal games on graphs: The game 0.33 on subdivided stars and bistars ☆

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A B S T R A C T

Octal games are a well-defined family of two-player games played on heaps of counters, in which the players remove alternately a certain number of counters from a heap, sometimes being allowed to split a heap into two nonempty heaps, until no counter can be removed anymore.

We extend the definition of octal games to play them on graphs: heaps are replaced by connected components and counters by vertices. Thus, playing an octal game on a path P_n is equivalent to playing the same octal game on a heap of n counters.

We study one of the simplest octal games, called 0.33, in which the players can remove one vertex or two adjacent vertices without disconnecting the graph. We study this game on trees and give a complete resolution of this game on subdivided stars and bistars.

1. Introduction

Combinatorial games are finite two-player games without chance, with perfect information and such that the last move alone determines which player wins the game. Since the information is perfect and the game finite, there is always a winning strategy for one of the players. A formal definition of combinatorial games and basic results will be given in Section 2. For more details, the interested reader can refer to [1], [2] or [3].

A well-known family of combinatorial games is the family of *subtraction games*, which are played on a heap of counters. A subtraction game is defined by a list of positive integers L and is denoted by $Sub(L)$. A player is allowed to remove k counters from the heap if and only if $k \in L$. The first player unable to play loses the game. For example, consider the game $Sub(\{1, 2\})$. In this game, both players take turns removing one or two counters from the heap, until the heap is empty. If the initial number of counters is a multiple of 3, then the second player has a winning strategy: by playing in such a way that the first player always gets a multiple of 3, he will take the last counter and win the game.

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A natural generalization of subtraction games is to allow the players to split a heap into two nonempty heaps after having removed counters. This defines a much larger class of games, called *octal games* [1]. An octal game is represented by an octal code which entirely defines its rules. As an example, $Sub(\{1,2\})$ is defined as **0.33**. A precise definition will be given in Section 2. Octal games have been extensively studied. One of the most important questions [4] is the periodicity of these games. Indeed, it seems that all finite octal games have a periodic behavior in the following sense: the set of initial numbers of counters for which the first player has a winning strategy is ultimately periodic. This is true for all subtraction games and for all finite octal games for which the study has been completed [5,1].

Octal games can also be played by placing counters in a row. Heaps are constituted by consecutive counters and only consecutive counters can be removed. According to this representation, it seems natural to play octal games on more complex structures like graphs. A position of the game is a graph and players remove vertices that induce a connected component which corresponds to consecutive counters. The idea to extend the notion of octal games to graphs was already suggested in [6]. However, to our knowledge, this idea has not been further developed. With our definition, playing the generalization of an octal game on a path is the same as playing the original octal game. In the special case of subtraction games, players have to keep the graph connected. As an example, playing **0.33** on a graph consists in removing one vertex or two adjacent vertices from the graph without disconnecting it.

This extension of octal games is in line with several take-away games on graphs such as Arc Kayles [7] and Grim [8]. However, it does not describe some other deletion games, such as the vertex and edge versions of the game geography [7,9], vertex and edge deletion games with parity rules, considered in [10] and [11], or scoring deletion games such as Le Pic arête [12].

We will first give in Section 2 basic definitions from combinatorial game theory as well as a formal definition of octal games on graphs. We then focus on the game **0.33** which is one of the simplest octal games, and to its study on trees. We first study subdivided stars in Section 3. We prove that paths can be reduced modulo 3 which leads to a complete resolution, in contrast with the related studies on subdivided stars of Node Kayles [6] and Arc Kayles [13]. In Section 4, we extend our results to subdivided bistars (i.e. trees with at most two vertices of degree at least 3) using a game operator similar to the sum of games. Unfortunately, these results cannot be extended to all trees and not even to caterpillars. In a forthcoming paper [14], some of our results are generalized to other subtraction games on subdivided stars.

2. Definitions

2.1. Basics of combinatorial game theory

Combinatorial games [1] are two-player games such that:

1. The two players play alternately.
2. There is no chance.
3. The game is finite (there are finitely many positions and no position can be encountered twice during the game).
4. The information is perfect.
5. The last move alone determines the winner.

In *normal* play, the player who plays the last move wins the game. In *misère* play, the player who plays the last move loses the game. *Impartial games* are combinatorial games where at each turn the moves are the same for both players. Hence the only distinction between the players is who plays the first move. In this paper, we will only consider impartial games in normal play.

Positions in impartial games have exactly two possible *outcomes*: either the first player has a winning strategy, or the second player has a winning strategy. If a game position falls into the first category, it is an $\mathcal{N}$*-position* (for $\mathcal{N}$ext player wins); otherwise, it is a $\mathcal{P}$*-position* (for $\mathcal{P}$revious player wins).

From a given position J of the game, the different positions that can be reached by playing a move from J are the *options* of J, and the set of options of J is denoted $\operatorname{opt}(J)$. If we know the outcomes of the positions in $\operatorname{opt}(J)$ we can deduce the outcome of J, using the following proposition:

Proposition 1. *Let J be a position of an impartial combinatorial game in normal play:*

- *If $\operatorname{opt}(J)=\emptyset$, then J is a $\mathcal{P}$-position.*
- *If there exists a $\mathcal{P}$-position J' in $\operatorname{opt}(J)$, then J is an $\mathcal{N}$-position: a winning move consists in playing from J to J'.*
- *If all the options of J are $\mathcal{N}$-positions, then J is a $\mathcal{P}$-position.*

Every position J of a combinatorial game can be viewed as a combinatorial game with J as the initial position. We therefore often consider positions as games. Some games can be described as the union of smaller game positions. In order to study them, we define the concept of the sum of games. Given two games J_1 and J_2, their *disjoint sum*, denoted by $J_1 + J_2$, is defined as the game where, at their turn, each player plays a legal move on either J_1 or J_2. Once J_1 (resp. J_2)

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