Doctopic: Algorithms, automata, complexity and games ARTICLE IN PRESS TCS:11651

Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

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Degree constrained 2-partitions of semicomplete digraphs ☆

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A R T I C L E   I N F O

Article history:
Received 14 November 2017
Received in revised form 6 June 2018
Accepted 15 June 2018
Available online xxxx
Communicated by R. Klasing

Keywords:
Semicomplete digraph
Tournament
2-partition
Minimum semi-degree
Minimum out-degree
Minimum in-degree
NP-complete
Digraphs of bounded independence number

A B S T R A C T

A 2-partition of a digraph D is a partition (V_1, V_2) of $V(D)$ into two disjoint non-empty sets V_1 and V_2 such that $V_1 \cup V_2 = V(D)$. A semicomplete digraph is a digraph with no pair of non-adjacent vertices. We consider the complexity of deciding whether a given semicomplete digraph has a 2-partition such that each part of the partition induces a (semicomplete) digraph with some specified property. In [4] and [5] Bang-Jensen, Cohen and Havet determined the complexity of 120 such 2-partition problems for general digraphs. Several of these problems are NP-complete for general digraphs and thus it is natural to ask whether this is still the case for well-structured classes of digraphs, such as semicomplete digraphs. This is the main topic of the paper. More specifically, we consider 2-partition problems where the set of properties are minimum out-, minimum in- or minimum semi-degree. Among other results we prove the following:

- For all integers $k_1, k_2 \geq 1$ and $k_1 + k_2 \geq 3$ it is NP-complete to decide whether a given digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i \rangle$ has out-degree at least k_i for $i = 1, 2$.
- For every fixed choice of integers $\alpha, k_1, k_2 \geq 1$ there exists a polynomial algorithm for deciding whether a given digraph of independence number at most α has a 2-partition (V_1, V_2) such that $D\langle V_i \rangle$ has out-degree at least k_i for $i = 1, 2$.
- For every fixed integer $k \geq 1$ there exists a polynomial algorithm for deciding whether a given semicomplete digraph has a 2-partition (V_1, V_2) such that $D\langle V_1 \rangle$ has out-degree at least one and $D\langle V_2 \rangle$ has in-degree at least k.
- It is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) such that $D\langle V_i \rangle$ is a strong tournament.

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1. Introduction

A 2-***partition*** of a (di)graph G is a partition (V_1, V_2) of $V(G)$ into two disjoint non-empty sets. Let $\mathcal{P}_1, \mathcal{P}_2$ be two graph properties, then a $(\mathcal{P}_1, \mathcal{P}_2)$-***partition*** is a 2-partition (V_1, V_2) where V_1 induces a graph with property $\mathcal{P}_1$ and V_2 a graph with property $\mathcal{P}_2$. For example a $(\delta^+ \geq k, \delta^+ \geq k)$-partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least k. It is natural to ask when properties such as high (edge)-connectivity or

☆ Research supported by the Danish research council under grant number DFF-1323-00178 and grant number DFF-7014-00037B.
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https://doi.org/10.1016/j.tcs.2018.06.028
0304-3975/© 2018 Published by Elsevier B.V.

Please cite this article in press as: J. Bang-Jensen, T.M. Christiansen, Degree constrained 2-partitions of semicomplete digraphs, Theoret. Comput. Sci. (2018), https://doi.org/10.1016/j.tcs.2018.06.028

minimum degree can be maintained by both parts of some 2-partition of a (di)graph G. As an example of this, Alon [1] and independently Stiebitz [14] posed the following problem.

Problem 1.1. *[1,14] Does there exist a function $h(k_1,k_2)$ such that every digraph $D=(V,A)$ with minimum out-degree $h(k_1,k_2)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition?*

It is easy to see that the answer to Problem 1.1 is yes if and only if there exists a function $h'(k_1,k_2) \geq k_1+k_2+1$ such that every digraph with minimum out-degree $h'(k_1,k_2)$ contains disjoint induced subdigraphs D_1, D_2 such that D_1 has minimum out-degree at least k_1 and D_2 has minimum out-degree at least k_2. The lower bound comes from the complete digraph on k_1+k_2+1 vertices in which every vertex has out-degree k_1+k_2. Clearly this does not have the desired 2-partition since there will be too few vertices in one of the sets of any 2-partition.

It is known that high minimum out-degree guarantees many disjoint cycles in a digraph [1,15], in particular minimum out-degree 3 is enough to guarantee two disjoint cycles. Using this, one can easily show that $h(1,1)=3$ (see the beginning of Section 3). Already the existence of $h(1,2)$ is open and we will show in Theorem 3.1 that it is NP-complete to decide if a digraph has a $(\delta^+ \geq 1, \delta^+ \geq 2)$-partition.

The following two problems are natural variations of Problem 1.1. For definitions of semi-degree etc, see the next section.

Problem 1.2. *Do there exist functions $w_1(k_1,k_2)$, $w_2(k_1,k_2)$ such that every digraph $D=(V,A)$ with minimum out-degree $\delta^+(D) \geq w_1(k_1,k_2)$ and minimum in-degree $\delta^-(D) \geq w_2(k_1,k_2)$ has a $(\delta^+ \geq k_1, \delta^- \geq k_2)$-partition.*

Problem 1.3. *Does there exist a function $z(k_1,k_2)$ such that every digraph $D=(V,A)$ with minimum semi-degree $\delta^0(D) \geq z(k_1,k_2)$ has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition.*

In [10] Lichiardopol answered Problems 1.1 to 1.3 in the affirmative for tournaments. It can easily be seen that his proofs can be generalized to semicomplete digraphs.

Theorem 1.4. *[10] Let T be a tournament (semicomplete digraph) with minimum out-degree at least $\frac{k_1^2+3k_1+2}{2}+k_2$, then T has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition.*

Theorem 1.5. *[10] Let T be a tournament (semicomplete digraph) with minimum semi-degree at least $k_1^2+3k_1+2+k_2$, then T has a $(\delta^0 \geq k_1, \delta^0 \geq k_2)$-partition.*

In [4,5] Bang-Jensen, Havet and Cohen determined the complexity of 120 partition problems for general digraphs. Among these are the following.

Theorem 1.6. *[5] The following 3 decision problems are NP-complete for general digraphs:*

- *deciding whether D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition,*
- *deciding whether D has a $(\delta^0 \geq 1, \delta^- \geq 1)$-partition and*
- *deciding whether D has a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition.*

We show in the beginning of Section 3 that the $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition problem is polynomially solvable for general digraphs. From these results two natural questions emerge. What is the complexity of the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem when $k_1+k_2 \geq 3$ and what is the complexity of the three problems in Theorem 1.6 if we restrict the input to a well-structured class of digraphs such as semicomplete digraphs? This paper will answer these questions as well as several related ones. In Section 3 we prove that as soon as $k_1+k_2 \geq 3$ the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem becomes NP-complete for general digraphs. Then we prove that for all fixed pairs of integers $k_1, k_2 \geq 1$ there exists a polynomial algorithm for the $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition problem for semicomplete digraphs. In Section 4 we prove that all three problems from Theorem 1.6 are polynomially solvable for semicomplete digraphs and in Section 5 we prove that for every fixed integer $k \geq 1$ the $(\delta^+ \geq 1, \delta^- \geq k)$-problem is polynomially solvable for semicomplete digraphs. In Section 6 we prove that, even for semicomplete digraphs, if we require several properties for each part of a 2-partition, we may obtain problems that are NP-complete, by showing that it is NP-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1,V_2) so that $D\langle V_i \rangle$ is a stong tournament for $i=1,2$. Finally in Section 7 we conclude with some remarks and open problems. In particular we outline why one of our proofs generalizes to digraphs of bounded independence number.

2. Notation, definitions and preliminary results

Notation follows [3] all digraphs considered have neither loops nor parallel arcs. We use the shorthand notation $[k]$ for the set $\{1,2,\dots,k\}$ and $[i,k]$ for the set $\{i,i+1,\dots,k\}$. Let $D=(V,A)$ be a digraph with vertex set V and arc set

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