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Burning number of graph products

Dieter Mitsche [a], Paweł Prałat [b,*,1], Elham Roshanbin [c]

[a] *Université de Nice Sophia-Antipolis, Laboratoire J-A Dieudonné, Parc Valrose, 06108 Nice cedex 02, France*
[b] *Department of Mathematics, Ryerson University, Toronto, ON, Canada*
[c] *Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, B3H 3J5, Canada*

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A B S T R A C T

Graph burning is a deterministic discrete time graph process that can be interpreted as a model for the spread of influence in social networks. The burning number of a graph is the minimum number of steps in a graph burning process for that graph. In this paper, we consider the burning number of graph products. We find some general bounds on the burning number of the Cartesian product and the strong product of graphs. In particular, we determine the asymptotic value of the burning number of hypercube graphs and we present a conjecture for its exact value. We also find the asymptotic value of the burning number of the strong grids, and using that we obtain a lower bound on the burning number of the strong product of graphs in terms of their diameters. Finally, we consider the burning number of the lexicographic product of graphs and we find a characterization for that.

1. Introduction

Graph burning is a graph process that models the spread of influence in social networks and was introduced in [3,4,8]. Here is the definition of this process which is defined on the node set of a simple finite graph. There are discrete time-steps (or rounds) and initially all nodes are unburned. In the first round, we choose one node that catches fire. At the beginning of every round t ($t \geq 2$), the fire spreads from the set of burning nodes to their unburned neighbours. Then we choose one node and start the fire there, unless the node is already on fire. (Of course, choosing a node that is already on fire is usually suboptimal but we allow this to avoid complications with situations in which no unburned node is available.) Throughout the process, each node is either *burned* or *unburned*. Once a node is burned it remains in that state until the end of the process. The process ends at the end of round T when all nodes are burning.

Suppose that we burn a graph G in k steps in a burning process. For $1 \leq i \leq k$, the node x_i that we choose to burn directly in the i-th step of this process is called the i-th *fire source*. The sequence $(x_1, \dots, x_k)$ is called a *burning sequence* for G. The *burning number* of a graph G, written by $b(G)$, is the length of a shortest burning sequence for G. Such a burning sequence is called an *optimum* burning sequence for G. For example, it is easy to see that $b(C_4) = 2$; the sequence (v_1, v_3) is an optimum burning sequence for C_4, as shown in Fig. 1. The red nodes and edges demonstrates the fire spread from v_1 and the blue node is the fire started at v_3. The burning number can be used as a measure for the speed of spreading fire on the node set of graphs.

* Corresponding author.
E-mail addresses: dmitsche@unice.fr (D. Mitsche), pralat@ryerson.ca (P. Prałat), e.roshanbin@dal.Ca (E. Roshanbin).

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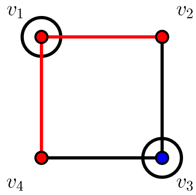

Fig. 1. An optimum burning sequence for C_4.

Given two graphs G and H, one can create a new graph on the node set $V(G) \times V(H)$. There are several different ways to define the connections (or the edges) of such a graph, and they have been studied well in the theory of graphs; see [6]. Since the burning number is a relatively new parameter, it is natural to consider the burning number of graph products. Several facts and bounds on the burning number of graphs are given in [1,4,8]. It is shown in [2,8] that the graph burning problem is **NP**-complete even for trees and path-forests. Some probabilistic results on the burning number of graphs, and some random variations of graph burning are presented in [7,8]. In this paper, we consider the burning number of graph products and its relation to the burning number of the initial graphs.

2. Preliminaries

We first present some terminology, and then we review some known facts about graph burning and the burning number that are needed throughout the paper. We denote a path of order n by P_n. For every pair of nodes u and v in a graph G, the number of edges in a shortest path between u and v in G is called the *distance* between u and v and is denoted by $d_G(u, v)$ (to emphasize the graph), or by $d(u, v)$ (for short, when there is no confusion). For a node v in G, the *eccentricity* of v is defined as $\max\{d(v, u) : u \in V(G)\}$. The *radius* of G, denoted by $\mathrm{rad}(G)$, is the minimum eccentricity of a node in G. The *centre* of G is the set of the nodes in G with minimum eccentricity. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity over the node set of G. For a positive integer k, the *k-th closed neighbourhood* of node v, denoted by $N_k[v]$, is defined to be the set $\{u \in V(G) : d(u, v) \le k\}$. We sometimes use the notation $N_k^G[v]$ to emphasize that we consider the k-th closed neighbourhood of node v in a specified graph G.

The *Cartesian product* of two graphs G and H, denoted by $G\square H$, is the graph with node set $V(G) \times V(H)$, in which two nodes (u_1, v_1) and (u_2, v_2) are adjacent if and only if, either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$. The *strong product* of two graphs G and H, denoted by $G \boxtimes H$, is the graph with node set $V(G) \times V(H)$, in which two nodes (u_1, v_1) and (u_2, v_2) are adjacent if and only if $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$. It is known that $d_{G\boxtimes H}((u_1, v_1), (u_2, v_2)) = \max\{d_G(u_1, u_2), d_H(v_1, v_2)\}$ (see, for example, [6]). By definition, we get immediately that $G\square H \subseteq G \boxtimes H$.

The *lexicographic product* of two graphs G and H, denoted by $G \circ H$, is the graph with node set $V(G) \times V(H)$, in which two nodes (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. In other words, $G \circ H$ is isomorphic to the graph that is constructed by replacing each node u_i in G by a copy of H, called H_i, and then adding all edges uv, where $u \in V(H_i)$, $v \in V(H_j)$, and u_iu_j is an edge in G. Namely, for $1 \le i \le |V(G)|$, $V(H_i) = \{(u_i, v) : v \in V(H)\}$. If $d_G(u_i, u_l)$ and $d_H(v_j, v_s)$ are finite (that is, u_i, u_j belong to the same connected component of G and v_j, v_s are in the same component of H), then for the nodes (u_i, v_j) and (u_l, v_s) in $G \circ H$, the following holds: if $u_i \ne u_l$, then

$$d_{G\circ H}\big((u_i, v_j), (u_l, v_s)\big) = d_G(u_i, u_l);$$

if $u_i = u_l$ and $v_j \ne v_s$, then

$$d_{G\circ H}\big((u_i, v_j), (u_l, v_s)\big) = \min\{2, d_H(v_j, v_s)\}.$$

For more on graph products see, for example, [6].

A subgraph H of a graph G is called an *isometric subgraph* if for every pair of nodes u and v in H, we have that $d_H(u, v) = d_G(u, v)$. For example, a subtree of a tree is an isometric subgraph. Also, if G is a connected graph and P is a shortest path connecting two nodes of G, then P is an isometric subgraph of G. For two functions $f(n)$ and $g(n)$, we write $f(n) \sim g(n)$ and we say that f is *asymptotic* to g (see [5] for more on asymptotic notations), if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Here are some facts about the burning number from [3,8] that we need for proving the results in this paper. From the definition of the burning process we can easily conclude the following lemma which is equivalent to Lemma 1 in [4].

Lemma 1. *A sequence $(x_1, x_2, \ldots, x_k)$ forms a burning sequence for a graph G if and only if*

$$N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \ldots \cup N_0[x_k] = V(G) \qquad (1)$$