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A refined mean field approximation of synchronous discrete-time population models

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ABSTRACT

Mean field approximation is a popular method to study the behaviour of stochastic models composed of a large number of interacting objects. When the objects are asynchronous, the mean field approximation of a population model can be expressed as an ordinary differential equation. When the objects are (clock-) synchronous the mean field approximation is a discrete time dynamical system. We focus on the latter.

We study the accuracy of mean field approximation when this approximation is a discrete-time dynamical system. We extend a result that was shown for the continuous time case and we prove that expected performance indicators estimated by mean field approximation are $O(1/N)$-accurate. We provide simple expressions to effectively compute the asymptotic error of mean field approximation, for finite time-horizon and steady-state, and we use this computed error to propose what we call a *refined* mean field approximation. We show, by using a few numerical examples, that this technique improves the quality of approximation compared to the classical mean field approximation, especially for relatively small population sizes.

1. Introduction

Stochastic models are often used to model and analyse the performance of computer (and many other) systems. A particularly rich and popular class of models is given by stochastic population models. These have been used, for instance, to model biological systems [1], epidemic spreading [2] or queuing networks [3]. These systems are composed of a set of homogeneous objects interacting with one another. These models have a high expressive power, but an exact analysis of any such a model is often computationally prohibitive when the number of objects of the system grows. This results in the need for approximation techniques.

A popular technique is to use mean field approximation. The idea behind mean field approximation is to replace the study of the original stochastic system by the one of a, much simpler, deterministic dynamical system. The success of mean field approximation can be explained by multiple factors: (a) it is fast – many models can be solved in closed form [3–6] or easily solved numerically [7–9] – (b) it is proven to be asymptotically optimal as the number of objects in the system goes to infinity [10–14]; and (c) it is often very accurate also for systems of moderate size, composed of $N \approx 100$ objects.

The mean field approximation of a given model is constructed by considering the limit of the original stochastic model as the number of objects N goes to infinity. There can be two types of limits. The first type arises when the dynamics of the objects are asynchronous. In this case the mean field approximation is given by a continuous time dynamical system (often

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a system of ordinary differential equations) — this is the most studied case *e.g.* [10,12,14]. The second type arises when the objects are synchronous. In this case the mean field approximation is a discrete time dynamical system [11,15,16]. We focus on the latter.

Contributions. Our main contribution is an extension to (synchronous) DTMC population models of the results proposed in [17] for (asynchronous) CTMC population models, thus providing a new approximation technique that is significantly more accurate than classical mean field approximation, especially for relatively small systems. Our results apply to the classical model of [11,15,18]. We prove our result for the transient and the steady state dynamics. Moreover, it retains an interesting feature of mean field approximation by being computationally non-intensive.

More precisely, if $M_i^{(N)}(t)$ denotes the proportion of objects in a state i at time t, then the classical result of [11] states that, as N grows large, if the vector $M^{(N)}(0)$ converges almost surely to m, for some vector m, then the vector $M^{(N)}(t)$ converges almost surely to a deterministic quantity $\mu(t)$ that satisfies a recurrence equation of the form $\mu(t+1) = \mu(t)\mathbf{K}(\mu(t))$ with $\mu(0) = m$. We show that, for any twice differentiable function h, there exists a constant $V_{t,h}$ such that

$$\lim_{N\to\infty} N(\mathbb{E}\left[h(M_t^{(N)})\right] - h(\mu(t))) = V_{t,h}. \quad (1)$$

We provide an algorithm to compute the constant $V_{t,h}$ by a linear dynamical system that involves the first and second derivative of the functions $m \mapsto m\mathbf{K}(m)$ and h. We also show that if the function $m \mapsto m\mathbf{K}(m)$ has a unique fixed point $\mu(\infty)$ that is globally exponentially stable, then the same result holds for the steady-state: in this case, $V_{\infty,h} = \lim_{t\to\infty} V_{t,h}$ exists and can be expressed as the solution of a discrete-time Lyapunov equation that involves the first and second derivative of $m \mapsto m\mathbf{K}(m)$ and h evaluated at the point $\mu(\infty)$.

By using these results, we define a quantity $h(\mu(t)) + V_{t,h}/N$ that we call the *refined mean field* approximation. As opposed to the classical mean field approximation, this approximation depends on the system size N. We illustrate our theoretical results with four different examples. While these examples all show that our refined model is clearly more accurate than the classical approximations, they illustrate different characteristics. The first two examples are cases where the dynamical system has a unique exponentially stable attractor. In these examples, refined mean field provides performance estimates that are extremely accurate (the typical error between $M^{(N)}$ and μ is less than 1% for $N = 10$). The third example is different as it is a case when the stochastic system has two absorbing states. In this case, the refined mean field is still more accurate than the classical mean field approximation but remains far from the exact values for $N = 10$. It is only for larger values of N that the refined mean field provides a very accurate estimate. Finally, the fourth example is a case where the mean field approximation has a unique attractor that is not exponentially stable. We observe that in this case the refined approximation provides an accurate approximation of $\mathbb{E}[M^{(N)}(t)]$ for small values of t but fails to predict correctly what happens when t is large compared to N. In fact in this case, one cannot refine the steady-state expectation by a term in $O(1/N)$ because the convergence is only in $O(1/\sqrt{N})$ in this case.

This suggests that, when using a mean field or refined mean field approximation, one has to be careful: the approximations of a system with more than one stable equilibrium or a unique but non-exponentially stable equilibrium is likely to be inaccurate for small values of N, even when one focuses on the transient behaviour.

Related work. Our results extend the recent results of [17]. The authors of [17] study the steady-state of stochastic models that have a continuous-time mean field approximation. They show that Eq. (1) is true in this case and provide a numerical algorithm to compute the constant. Our paper has two theoretical contributions with respect to [17]: First we show that the results also hold for models that have a discrete-time mean field approximation, and second we show how to derive these equations for the transient and steady state regimes of such systems. This means that our results remain in the realm of discrete-time models whereas in [17] it is shown how some discrete-time models can be transformed into density-dependent (continuous time) population models by replacing time steps with steps that last for a random time that is exponentially distributed with mean $1/N$, where N is the population size. The resulting continuous time model can then be analysed using the approximation techniques for CTMC population models discussed in [17].

The results of [17] and the one of the current paper follow from a series of recent results concerning the rate of convergence of stochastic models to their mean field approximation [19–22]. The key idea behind these works is to study the convergence of the generator of the stochastic processes to the one of its mean field approximation and to use this convergence rate to obtain a bound on Eq. (1). For the steady-state regime, this is made possible by using Stein's methods [23–25]. Note that the approach taken in the current paper is fundamentally different from the one that is usually used to obtain convergence rates, like [13,15,26] in which the authors focus on sample path convergence and obtain bounds on the convergence of the expected distance $\mathbb{E}[\|M^{(N)} - \mu\|]$ between the stochastic system and its mean field approximation. When focusing on sample path convergence, the refinement of the mean field approximation would be to consider an additive term of $1/\sqrt{N}$ times a Gaussian noise as for example in [13] and not a $1/N$ term as in this paper.

This paper and the simulations it contained are fully reproducible: https://github.com/ngast/RefinedMeanField_SynchronousPopulation.

Outline. The rest of the paper is organised as follows. In Section 2, we introduce the model that we study. In Section 3 we provide the main results and in particular Theorem 1. In Sections 4–7, we provide a few numerical examples that demonstrate the accuracy of the refined mean field approximation and its limits. Finally, we conclude in Section 8.

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