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Higher order FEM for the obstacle problem of the p-Laplacian—A variational inequality approach

Lothar Banz [a,*], Bishnu P. Lamichhane [b], Ernst P. Stephan [c]

[a] Department of Mathematics, University of Salzburg, Hellbrunner Straße 34, 5020 Salzburg, Austria
[b] School of Mathematical & Physical Sciences, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia
[c] Institute of Applied Mathematics, Leibniz University Hannover, Welfengarten 1, 30167 Hannover, Germany

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ABSTRACT

We consider higher order finite element discretizations of a nonlinear variational inequality formulation arising from an obstacle problem with the p-Laplacian differential operator for $p \in (1, \infty)$. We prove an a priori error estimate and convergence rates with respect to the mesh size h and in the polynomial degree q under assumed regularity. Moreover, we derive a general a posteriori error estimate which is valid for any uniformly bounded sequence of finite element functions. All our results contain the known results for the linear case of $p = 2$. We present numerical results on the improved convergence rates of adaptive schemes (mesh size adaptivity with and without polynomial degree adaptation) for the singular case of $p = 1.5$ and for the degenerated case of $p = 3$.

1. Introduction

In 2012 a new strand of literature has arisen in which steady, shallow ice sheet flows like the Greenland ice sheet or glaciers in general are mathematically modeled by a (generalized) p-Laplacian obstacle problem [1]. The ice sheet itself is modeled by the singular case ($1 < p < 2$) of the p-Laplacian operator $-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ which in conjunction with the bedrock as an obstacle leads to an obstacle problem of the p-Laplacian. In [2], the solution of the p-Laplacian obstacle problem for $2 < p < \infty$ is interpreted as limit value of a discrete tug-of-war game with noise.

The numerical analysis and the numeric itself of the p-Laplacian obstacle problem face several challenges. Firstly, the p-Laplacian is not uniformly elliptic. For $1 < p < 2$ it becomes singular where the gradient of u vanishes, and for $2 < p < \infty$ it degenerates at these points. A break-through in the numerical analysis of the p-Laplacian variational equation was the introduction of a quasi-norm, cf. [3–6] among others, which allows to connect the weak form of the operator with a norm, see here Lemma 2 for that. Secondly, the non-linear variational inequality of the p-Laplacian obstacle problem is, as any obstacle problem, characterized by a free boundary problem. The domain Ω can be separated into two subsets, a contact subset, in which u equals the obstacle, and a non-contact subset, in which u is the solution of a simpler p-Laplacian equation. Finding the free boundary which separates these two subsets of Ω is the actual numerical task at hand. Moreover, the solution exhibits a singularity at the a priori unknown location of the free boundary. Thus efficient numerical schemes require automatic mesh size h (and ideally also polynomial degree q) adaptation based on an a posteriori error estimate. We refer to [4,5,7–9] among many others for the p-Laplacian equation, to [10–14] among many others for the Laplacian obstacle problem and to our Theorems 11 and 13 for the p-Laplacian obstacle problem. Thirdly, the residual of the non-linear variational inequality

* Corresponding author.
E-mail addresses: lothar.banz@sbg.ac.at (L. Banz), Bishnu.Lamichhane@newcastle.edu.au (B.P. Lamichhane), stephan@ifam.uni-hannover.de (E.P. Stephan).

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contains valuable information of the actual contact of u with the rigid obstacle and is thus used in our a posteriori error estimates, cf. Theorem 11. As the p-Laplacian operator becomes singular or degenerates where the gradient of u vanishes, which, e.g., for constant obstacles is the entire actual contact set, we no longer have a one-to-one correspondence between the residual and the primal variable u, see, e.g., Lemma 4 and Section 4.

Most finite element schemes for the p-Laplacian equation are restricted to the conforming lowest order piecewise linear basis functions, as in general high regularity may not be achievable even with smooth data [6,15]. Consequently, one should resort to adaptive schemes, in which the mesh size is automatically and locally refined, when striving for higher and potentially optimal algebraic convergence rates. Obviously, the benefit of adaptivity gets bigger with higher polynomial degree. If also the polynomial degree is increased locally where the solution is locally regular, even exponential convergence may be achievable as for uniformly elliptic equations, see, e.g., [16]. In principle the same holds for obstacle problems, however, there the singular points, the free boundary, form a $(d-1)$-dimensional manifold which can only be resolved poorly by isotropic refinements. As a consequence isotropic h- and hq-adaptive schemes for obstacle problems have an upper bound on their algebraic convergence rate. The risk of choosing the polynomial degree too high or too small in h-adaptive schemes can be circumvented by choosing an hq-adaptive scheme. Non-isotropic refinement for obstacle problems is still an open question.

Any reasonable numerical scheme must be guaranteed to converge under assumed regularity. The pioneering work [17] of Falk to prove convergence rates of the linear h-version for the Laplacian obstacle problem has been extended to the q-version of non-linear but uniformly elliptic variational inequality in an $L^2(\Omega)$-setting in [18] and is further extended to the p-Laplacian obstacle problem in our Theorems 8 and 9. The estimation of the non-conformity of the discrete trial and test convex set K_{hq}, arising from the discretization of the non-penetration constraint, significantly reduces the proven convergence rates. In the linear case, the discretization becomes essentially conforming and thus better (optimal) convergence rates can be proven, cf. [17] and Corollary 10.

The rest of the paper is structured as follows: We placed several general but fundamental estimates in Appendix. In Section 2 a variational inequality formulation of the p-Laplacian obstacle problem is analyzed. The mentioned weak formulation is discretized within a higher order finite element setting in Section 3. That section also includes the a priori error estimates. The a posteriori error analysis is placed in Section 4. Numerical results on the improved convergence rate of h-adaptive schemes and in particular of hq-adaptive schemes for $p=1.5$ and $p=3$ on an L-shape domain are presented in Section 5.

Notation: C, C' and such denote generic constants which may take different values at different positions. A prime at a variable p, q or k denotes the conjugated Hölder exponent, i.e., $p'=p/(p-1)$. Here,

$$W^{k,p}(\Omega)=\left\{v\in L^p(\Omega)\mid D^\alpha v\in L^p(\Omega)\,\forall\,|\alpha|\le k\right\}$$

denotes the Sobolev space equipped with the norm $\|v\|^p_{W^{k,p}(\Omega)}=\sum_{|\alpha|\le k}\|D^\alpha v\|^p_{L^p(\Omega)}$ and the semi-norm $|v|^p_{W^{k,p}(\Omega)}=\sum_{|\alpha|=k}\|D^\alpha v\|^p_{L^p(\Omega)}$ for any $k\in\mathbb{N}_0$ and $p\ge 1$. Moreover, we set $W^{k,p}_0(\Omega)=\left\{v\in W^{k,p}(\Omega)\mid v|_{\partial\Omega}=0\right\}$, and $W^{-k,p'}(\Omega)$ is its dual space. As usual for degenerated problems we utilize the quasi-norm (here specifically for the p-Laplacian)

$$|v|^2_{(1,w,p)}=\int_\Omega(|\nabla w|+|\nabla v|)^{p-2}|\nabla v|^2\,dx \tag{1}$$

for any $w\in W^{1,p}(\Omega)$, see, e.g., [4–6,19]. We find it useful to define a dual norm to the quasi-norm $|\cdot|_{(1,w,p)}$. For $\mu\in W^{-1,p'}(\Omega)$ we set

$$\|\mu\|_{(-1,w,p')}:=\sup_{v\in W^{1,p}_0(\Omega)\setminus\{0\}}\frac{\langle\mu,v\rangle}{|v|_{(1,w,p)}} \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. From its definition we have that $\|\cdot\|_{(-1,w,p')}$ does satisfy the absolute homogeneity, triangle inequality and positive-definiteness axioms of norms and, moreover, there holds $\langle\mu,v\rangle\le\|\mu\|_{(-1,w,p')}|v|_{(1,w,p)}$ for all $v\in W^{1,p}_0(\Omega)$ and $\mu\in W^{-1,p'}(\Omega)$.

2. The *p*-Laplacian obstacle problem

Let $\Omega\subset\mathbb{R}^2$ be an open, bounded, polygonal domain with boundary $\Gamma:=\partial\Omega$. Moreover, let $1<p<\infty$, $p'=p/(p-1)$ and data $f\in L^{p'}(\Omega)$, $\chi\in W^{1,p}(\Omega)$, with trace $\chi|_\Gamma=0$ for the ease of presenting the mixed formulation, be given. We consider the p-Laplacian obstacle problem of finding a suitable function u such that

$$-\Delta_p u:=-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\ge f,\quad u\ge\chi,\quad\left(-\Delta_p u-f\right)(u-\chi)=0\quad\text{in }\Omega\quad\text{and}\quad u=0\quad\text{on }\Gamma. \tag{3}$$

Its standard variational inequality formulation is to find $u\in K$ such that

$$a(u;u,v-u)\ge\langle f,v-u\rangle\quad\forall\,v\in K, \tag{4}$$

where

$$K:=\left\{v\in W^{1,p}_0(\Omega)\mid v\ge\chi\text{ a.e. in }\Omega\right\} \tag{5}$$

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