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A note on exponential Rosenbrock–Euler method for the finite element discretization of a semilinear parabolic partial differential equation

Jean Daniel Mukam [a], Antoine Tambue [b,c,d,*]

[a] Fakultät für Mathematik, Technische Universität Chemnitz, 09126 Chemnitz, Germany
[b] Department of Computing Mathematics and Physics, Western Norway University of Applied Sciences, Inndalsveien 28, 5063 Bergen, Norway
[c] Center for Research in Computational and Applied Mechanics (CERECAM), and Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, South Africa
[d] The African Institute for Mathematical Sciences(AIMS) of South Africa, 6-8 Melrose Road, Muizenberg 7945, South Africa

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ABSTRACT

In this paper, we consider the numerical approximation of a general second order semilinear parabolic partial differential equation. Equations of this type arise in many contexts, such as transport in porous media. Using finite element method for space discretization and the exponential Rosenbrock–Euler method for time discretization, we provide a convergence proof in space and time under only the standard Lipschitz condition of the nonlinear part, for both smooth and nonsmooth initial solutions. This is in contrast to restrictive assumptions made in the literature, where the authors have considered only approximation in time so far in their convergence proofs. The main result reveals how the convergence orders in both space and time depend heavily on the regularity of the initial data. In particular, the method achieves optimal convergence order $\mathcal{O}\left(h^2 + \Delta t^2\right)$ when the initial data belongs to the domain of the linear operator. Numerical simulations to sustain our theoretical result are provided.

1. Introduction

We consider the following abstract Cauchy problem with boundary conditions

$$\frac{du(t)}{dt} = Au(t) + F(u(t)), \quad u(0) = u_0, \quad t \in (0, T], \quad T > 0, \tag{1}$$

on the Hilbert space $H = L^2(\Lambda)$, where Λ is an open subset of $\mathbb{R}^d$ $(d = 1, 2, 3)$, which is supposed to be a convex polygon or has a smooth boundary. The linear operator $A : \mathcal{D}(A) \subset H \longrightarrow H$ is negative, not necessarily self adjoint and generates an analytic semigroup $S(t) := e^{At}$, $t \geq 0$. Without loss of generality, the nonlinear function $F : H \longrightarrow H$ is assumed to be autonomous. Our main focus will be on the case where A is a general second order elliptic operator. Under some technical conditions (see e.g. [1,2]), it is well known that the mild solution of (1) is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F\left(u(s)\right) ds, \quad t \in [0, T]. \tag{2}$$

* Corresponding author at: Department of Computing Mathematics and Physics, Western Norway University of Applied Sciences, Inndalsveien 28, 5063 Bergen, Norway.
E-mail addresses: jean.d.mukam@aims-senegal.org (J.D. Mukam), antonio@aims.ac.za (A. Tambue).

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In general, it is hard to find the exact solutions of many PDEs. Numerical approximations are currently the only important tools to approximate the solutions. Approximations are done at two levels, spatial approximation and temporal approximation. The finite element [3], finite volume [4], finite difference methods are mostly used for space discretization of the problem (1), while explicit, semi implicit and fully implicit methods are usually used for time discretization. References about standard discretization methods for (1) can be found in [4]. Due to the time step size constraints, fully implicit schemes are more popular for the time discretization for quite a long time compared to explicit Euler schemes. However, implicit schemes need at each time step a solution of large systems of nonlinear equations. This can be the bottleneck in computations when dealing with realistic problems. In recent years, exponential integrators have become an attractive alternative in many evolution equations [4–9]. Most exponential integrators analyzed early in the literature [5,7,8] were bounded on the nonlinear problem as in (1) where the linear part A and the nonlinear function F are explicitly known a priori. Such approach is justified in situations where the nonlinear function F is small. Due to the fact that in more realistic applications the nonlinear function F can be stronger,[1] Exponential Rosenbrock-Type methods have been proposed in [10,11], where at every time step, the Jacobian of F is added to the linear operator A. The lower order of them, called Exponential Rosenbrock–Euler method (EREM) has been proved to be efficient in various applications [9,12]. For smooth initial solutions, this method is well known to be second order convergence in time [10,11] and have good stability properties in the stochastic context [13]. However in many applications initial solutions are not always smooth. Typical examples are option pricing in finance or reaction diffusion advection with discontinuous initial solution. We refer to [14–20] for standard numerical technique with nonsmooth initial data. Recently exponential Rosenbrock–Euler with nonsmooth initial solution was analyzed in [21,22] under the additional hypothesis [21,22, Assumption 1]. Furthermore, to the best of our knowledge, only convergence in time is investigated for smooth or nonsmooth initial solution in all existing Exponential Rosenbrock-Type methods.

The goal of this paper is to provide a rigorous convergence proof of EREM in space and time for both smooth and nonsmooth initial solutions under more relaxed conditions than those used in [21,22]. Indeed only the standard Lipschitz condition of the nonlinear part is used in our convergence analysis and optimal convergence orders in space and time are achieved. In fact the method achieves convergence orders of $\mathcal{O}\left(h^{\beta}+\Delta t^{1+\beta/2}\right)$, where β is the regularity parameter of the initial data (see Assumption 2.1). Note that when dealing with space discretization, more novel and careful estimates need to be derived. This is because the constant appearing in the error estimate should not depend on the space discretization parameter h. The space discretization is performed using finite element method. Recent work in [4] can be used to obtain the similar convergence proof for finite volume method.

The paper is organized as follows. In Section 2, results about the well posedness are provided along with EREM scheme and the main result. The proof of the main result is presented in Section 3. In Section 4, we present some numerical simulations to sustain our theoretical result.

2. Mathematical setting and numerical method

2.1. Notations, setting and well posedness

Let us start by presenting briefly notations, the main function spaces and norms that will be used in this paper. We denote by $\|\cdot\|$ the norm associated to the inner product $(\cdot,\cdot)$ of the Hilbert space $H=L^2(\Lambda)$. The norms in the Sobolev spaces $H^m(\Lambda)$, $m\geqslant 0$ will be denoted by $\|.\|_m$. For a Hilbert space U we denote by $\|\cdot\|_U$ the norm of U, $L(U,H)$ the set of bounded linear operators from U to H. For ease of notation, we use $L(U,U)=:L(U)$. In the sequel, for convenience of presentation we take A to be a second-order operator as this simplifies the convergence proof. More precisely, we assume A to be given by

$$Au=\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(q_{ij}(x)\frac{\partial u}{\partial x_j}\right)-\sum_{i=1}^{d}q_i(x)\frac{\partial u}{\partial x_i}, \tag{3}$$

where $q_{ij}\in L^{\infty}(\Lambda)$, $q_i\in L^{\infty}(\Lambda)$. We assume that there is a constant $c_1>0$ such that

$$\sum_{i,j=1}^{d}q_{ij}(x)\xi_i\xi_j\geq c_1|\xi|^2,\quad \xi\in\mathbb{R}^d,\quad x\in\overline{\Omega}. \tag{4}$$

As in [23,24], we introduce two spaces $\mathbb{H}$ and V, such that $\mathbb{H}\subset V$, that depend on the choice of the boundary conditions for the domain of the operator A and the corresponding bilinear form. For example, for Dirichlet (or first-type) boundary conditions we take

$$V=\mathbb{H}=H_0^1(\Lambda)=\{v\in H^1(\Lambda):v=0\quad\text{on}\quad\partial\Lambda\}. \tag{5}$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition ($\alpha_0=0$), we take $V=H^1(\Lambda)$

$$\mathbb{H}=\{v\in H^2(\Lambda):\partial v/\partial v_A+\alpha_0 v=0,\quad\text{on}\quad\partial\Lambda\},\quad \alpha_0\in\mathbb{R}. \tag{6}$$

1 Typical examples are semilinear advection diffusion reaction equations with stiff reaction term.

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