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Stability and possible bifurcations for a Gent-Thomas elastic parallelepiped subject to dead-load surface tractions

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ABSTRACT

We study the equilibrium and the local stability for an incompressible elastic solid under arbitrary dead-load surface tractions on the boundary. We particularize the analysis to homogeneous deformations of a homogeneous, isotropic parallelepiped, finding necessary and sufficient algebraic local stability conditions consistent with the further requirement of the zero moment condition. We specialize the study for a uniform distribution of dead-load surface tractions $s > 0$ on two pairs of faces and $-s$ on the remaining two faces, and we find that two classes of equilibrium solutions may occur: symmetric and asymmetric solutions, respectively. For the symmetric solutions we also determine local stability inequalities. For the special case of a Gent-Thomas material we show that both equilibrium symmetric and asymmetric solutions may occur if the material parameters satisfy certain inequalities. Then, we completely describe the response of the parallelepiped in a loading process starting from the unloaded state for five ranges of the values of the material parameters. In particular, for one of these ranges we show that symmetric solutions are the unique locally stable homogenous equilibrium deformations until a critical value s_{cr} of the load; at s_{cr}, symmetric solutions lose their uniqueness, and a bifurcation into asymmetric solutions may occur.

1. Introduction

Stability and bifurcation topics play a fundamental role in the framework of the non-linear theory of elasticity, since they provide suitable mathematical tools for accurately modelling the complex behavior of engineering systems. At the fundamental level, bifurcation and stability theories deal with different general issues; bifurcation theory basically concerns the *non-uniqueness* of solutions, while stability theory regards the behavior of deformed bodies under disturbances. Indeed, in the literature one may find cases of bifurcations from a stable equilibrium solution without loss of stability, as in phenomena involving stable coexistent phases in stress-induced phase transformations (see, e.g., [1–3]), or situations characterized by a of loss of stability due to the presence of softening branches without the emergence of a new bifurcating mode. This common trend of separately treating bifurcation and stability issues is basically due to analytical complications, which occur especially within the context of three-dimensional non-linear elasticity.

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Among classical methods for studying the stability of equilibrium solutions, we recall the Hadamard energetic criterion of infinitesimal stability (cf. [4], §68 bis), which states that, during a loading process governed by a single loading parameter, locally stable deformations are those which render the second variation of the total potential energy functional greater or equal to zero among all the admissible deformations. In the terminology of Truesdell and Noll, the strict positivity of the Hadamard functional selects *superstable* (and consequently unique) equilibrium configurations, whereas *neutrally* stable deformations correspond to the smallest value of the loading parameter which first renders the Hadamard functional equal to zero, with a consequent loss of uniqueness for a primary superstable solution.

Since in most cases the determination of the sign of the Hadamard functional may be prohibitive, it may be very helpful developing a lower bound estimate for the Hadamard functional, with the major goal of seeking a lower bound estimate for the critical load, defined as the value of the loading parameter below which the Hadamard stability condition is definitely satisfied. In our previous papers [5,6] we have determined lower bound estimates of the Hadamard functional based on the Korn's inequality either for compressible or for incompressible hyperelastic solids. In particular, we believe that the procedure for the determination of the estimates proposed in [5], which has been successfully tested for the special bifurcation problem from a homogeneously deformed state contained in [7], may be also extended to the case of inhomogeneous deformations developed in [8].

The above considerations are useful for checking if a primary stable deformation becomes unstable and if a new bifurcation mode may emerge, but such methods do not help to assert if a bifurcation mode actually occurs or, in other words, if actually there are local branches of post-bifurcating solutions. We point out that, differently to what happens in the context of one-dimensional elasticity theory (see, for example, the paradigmatic case of the Euler beam, wherein the primary stable mode becomes unstable in correspondence of a bifurcation point and the new bifurcating mode is characterized by a stable post-critical response), within the three-dimensional non-linear elasticity framework there are very few results showing the same features of the one-dimensional cases. In this vein, we recall our previous paper [9], wherein a not common explicit example of a bifurcation from a locally stable primary deformation to a secondary bifurcation mode characterized by a locally stable post-critical branch has been developed.

Most of the above considerations characterize the core of the present paper, which is based upon [10], but it also contains additional significant results described below.

In Section 2 of the present work, we first recall necessary equilibrium and local stability conditions for an incompressible elastic solid subject on the whole boundary to an arbitrary distribution of dead-load surface tractions. Then, by particularizing our analysis to the case of homogeneous deformations for a homogeneous incompressible elastic body of arbitrary geometry, we find necessary and also *sufficient* algebraic local stability conditions which take into account the further requirement for the class of admissible deformations to satisfy the so-called *zero moment condition* (see [11]), in order to allow for the boundary traction to be always moment balanced.

In Section 3, we explicate the general necessary and also *sufficient* algebraic local stability conditions obtained in Section 2 for the case of an arbitrary homogeneous incompressible, *isotropic* elastic body. In particular, the employment of the zero-moment conditions allows to correlate the skew-symmetric part of the gradient of an admissible deformation to its symmetric part, with the consequence of reducing to five the algebraic local stability inequalities.

In Section 4, we study the equilibrium and the local stability of the homogeneous deformations of a homogeneous parallelepiped made of an arbitrary incompressible, isotropic elastic material, in absence of body forces, and subject to a *uniform* distribution of tensile dead-load surface tractions of amount $s > 0$ on two pairs of faces and of amount $-s$ (compressive) on the remaining two faces. This kind of traction boundary conditions should be useful in describing meaningful physical situations, such as the multiaxial loading experimental response of specimens subject to a equibiaxial tension accompanied by a uniaxial compressive force of the same amount. In particular, for a meaningful family of possible equilibrium solutions we find that two subclasses may occur, respectively named *symmetric* solutions, having two equal principal stretches, and *asymmetric* solutions, with all different principal stretches. For what concerns the symmetric solutions, we find that the local stability inequalities reduce form five to three.

Finally, in Section 5 we assume an explicit form for the strain energy function by considering a Gent-Thomas [12] incompressible, isotropic elastic parallelepiped. We first show that, under the uniform loading conditions introduced in Section 4, there is the actual possibility of having both equilibrium symmetric and asymmetric solutions if the material parameters satisfy certain inequalities. Then, by considering a loading process starting from the unloaded state, we completely describe the response of the parallelepiped as the material parameters vary within five ranges. In particular, we show that when the material constants belong to one of these intervals, the *symmetric* solutions are the unique locally stable homogenous equilibrium deformations until a critical value s_{cr} of the loading parameter is reached. In correspondence of s_{cr}, the *symmetric* solutions lose their uniqueness, and a bifurcation into equilibrium *asymmetric* solutions may occur. The analysis of the stability and the post-critical behavior of asymmetric deformations will represent the content of a forthcoming paper.

2. Equilibrium and stability for incompressible elastic solids under dead load surface tractions

Throughout the present work, $\Omega \subset \mathbb{R}^3$ will denote the open, bounded and connected region of space occupied by a hyperelastic, incompressible solid body in a given reference configuration. We will name $\mathbf{X} \in \Omega$ the points of Ω and $\partial\Omega$ the boundary of Ω, respectively, and for now we assume that Ω is possibly inhomogeneous.

Because of incompressibility, a deformation of Ω is described by a continuous isochoric function $\mathbf{f} : \Omega \to \mathbb{R}^3$ whose gradient field $\mathbf{F} := \nabla\mathbf{f}(\mathbf{X})$ belongs to

$$\mathbf{D} := \{\mathbf{F} \in \mathrm{Lin} | \det \mathbf{F} = 1\}, \tag{1}$$

where Lin denotes the set of the second order tensors. Furthermore, since the body Ω is also hyperelastic, its mechanical response is described for each $\mathbf{X} \in \Omega$ by a smooth function $\mathrm{W}(\cdot, \mathbf{X}) : \mathbf{D} \to \mathbb{R}$, namely the stored energy function per unit volume of Ω, whose common observer invariance assumption requires that, for any $\mathbf{F} \in \mathbf{D}$,

$$\mathrm{W}(\mathbf{QF}, \mathbf{X}) = \mathrm{W}(\mathbf{F}, \mathbf{X}) \tag{2}$$

for all proper orthogonal tensors[1] $\mathbf{Q}$. In addition, we assume that

$$\mathrm{W}(\mathbf{F}, \mathbf{X}) \geqslant \mathrm{W}(\mathbf{Q}, \mathbf{X}) = \mathrm{W}(\mathbf{I}, \mathbf{X}) = 0 \tag{3}$$

for all $\mathbf{F} \in \mathbf{D}$ and for all proper orthogonal tensors $\mathbf{Q}$, where $\mathbf{I}$ is the identity tensor. Finally, the tangent space $\mathbf{T}(\mathbf{F})$ of second order tensors associated with the constrained manifold $\mathbf{D}$ for each $\mathbf{F} \in \mathbf{D}$ is given by

$$\mathbf{T}(\mathbf{F}) := \{\mathbf{H} \in \mathrm{Lin} | \mathbf{H} \cdot \mathbf{F}^{-T} = 0\}. \tag{4}$$

The total Piola-Kirchhoff stress $\mathbf{S}(\mathbf{F}, \mathbf{X})$ is defined as follows:

$$\mathbf{S}(\mathbf{F}, \mathbf{X}) := \mathrm{W}_{\mathbf{F}}(\mathbf{F}, \mathbf{X}) - \mathrm{p}\mathbf{F}^{-T}, \tag{5}$$

where the first gradient $\mathrm{W}_{\mathbf{F}}(\cdot, \mathbf{X})$ of W evaluated at $\mathbf{F} \in \mathbf{D}$ defines the "active" part of the Piola-Kirchhoff stress, and $\mathrm{p} : \mathbf{X} \in \Omega \to \mathbb{R}$ denotes the Lagrangian multiplier field related to the incompressibility constraint. Clearly, $\mathbf{S}(\mathbf{F}, \mathbf{X})$ depends on the way W is extended to Lin^+. Moreover, the second gradient $\mathrm{W}_{\mathbf{FF}}(\cdot, \mathbf{X})$ evaluated at $\mathbf{F} \in \mathbf{D}$ defines the classical fourth order elasticity tensor from the reference configuration.

For what concerns the loading conditions, we consider *dead load surface tractions* $\hat{\mathbf{s}}(\mathbf{X})$ on the whole boundary $\partial\Omega$ and we neglect body forces. Thus, the total potential energy functional $\varepsilon(\mathbf{f})$ of the body in correspondence of a deformation $\mathbf{f}$ is given by

$$\varepsilon(\mathbf{f}) := \int_\Omega \mathrm{W}(\nabla\mathbf{f}, \mathbf{X}) - \int_{\partial\Omega} \hat{\mathbf{s}} \cdot \mathbf{f}, \tag{6}$$

where $\hat{\mathbf{s}}(\cdot) : \partial\Omega \to \mathbb{R}^3$ is a prescribed smooth map. According to the energy stability criterion, in presence of dead load tractions on $\partial\Omega$ a

[1] Following a usual practice in elasticity literature, henceforth we implicitly assume that $\mathrm{W}(\mathbf{F}, \mathbf{X})$ is arbitrarily extended to Lin^+, the set of second order tensors with positive determinant, and assume that this extended function is observer invariant and twice differentiable with respect to both of its arguments. This allows to perform derivatives of W which are not tangential to $\mathbf{D}$.

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