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Lattice Boltzmann model for the simulation of the wave equation in curvilinear coordinates

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ABSTRACT

Since its origins, lattice-Boltzmann methods have been restricted to rectangular coordinates, a fact which jeopardizes the applications to problems with cylindrical or spherical symmetries and complicates the implementations with complex geometries. However, M. Mendoza [1] recently proposed in his doctoral thesis a general procedure (based on Christoffel symbols) to construct lattice-Boltzmann models on curvilinear coordinates, which has shown very good results for hydrodynamics on cylindrical and spherical coordinates. In this work, we construct a lattice-Boltzmann model for the propagation of scalar waves in curvilinear coordinates, and we use it to determine the vibrational modes inside cylinders, trumpets and tori. The model correctly reproduces the theoretical expectations for the vibrational modes, and exemplifies the wide range of future applications of lattice-Boltzmann models on general curvilinear coordinates.

1. Introduction

Waves are present almost in every phenomenon, including optics, seismic prospection, mechanical oscillators and scalar acoustic of music instruments and concert halls. Many of those phenomena can be modeled just by considering scalar waves, and the relevant processes of reflection, refraction, interference and diffraction are more related to the wave equation itself than the mechanical laws governing the medium where the wave propagates. Thus, developing efficient computational methods to simulate scalar waves has been a subject of intense research [2–4].

Lattice-Boltzmann methods (LBM) are valuable alternatives for the simulation of partial differential equations. First introduced as mesoscopical models to simulate a wide variety of processes in fluid dynamics [5–10], they were later extended to more general systems, like electrodynamics [11], or even Quantum Mechanics [12]. Indeed, they can be considered nowdays as a general numerical scheme to solve differential equations that can be written as a set of conservation laws.

LBMs developed to reproduce the full Navier–Stokes equations (NSE) have also been used to simulate acoustic waves in compressible fluids [13–16]. Those models require high resolutions or high order schemes to be accurate and, therefore, they are very demanding in computational resources, although recent developments with regularized LMB schemes have reduced the computational cost and introduced better ways to include acoustic sources [17]. In contrast, as mentioned before, most

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wave phenomena can be described independently of the dynamics of the medium. Therefore, developing LBMs to simulate the scalar wave equation itself (without any fluid dynamics) would considerably reduce the computational demands and increase the stability. The first LBM to directly simulate the wave equation was proposed by Chopard, Luthi and Wagen [18]. They propose a different equilibrium distribution function and with a redefinition of the macroscopic variables, they can prove that in the macroscopic limit, the LBM reproduces the wave equation. Later, in 2000 Guangwu [19] proposed to redefine the first macroscopic momentum of the LBM as the temporal derivative of the wave pressure. This also leads to the pressure wave equation, but an additional integration step is needed.

Since they are easy to implement parallel on graphic cards, LBMs have gained the interest of a wide range of research areas and industrial applications. However, since most lattice-Boltzmann models assume a homogeneous and isotropic set of velocity vectors to move the information from node to node, the computational domain has been restricted to a rectangular array of cubic cells, forcing the use of staircase approximations on curved boundaries and imposing three-dimensional simulation domains for systems that, because of axial or spherical symmetries, were essentially two-dimensional, with an exorbitant increase in computational costs. This is also the case of simulating waves with lattice Boltzmann on acoustic systems. Symphonic and traditional instruments like violins, trumpets or drums, modern auditoriums, complex geological wells in seismic prospection and hearing organs like the Cochlea have too complex geometries to be properly described in Cartesian coordinates, asking for the need of new LBM for acoustics in generalized coordinates to take advantage of their versatility and parallel nature.

Various approaches have been proposed to overcome the Cartesian restriction in LBMs. The first 2D proposals require interpolation steps [20,21], sometimes combined with complex grid refinements [22]. Later developments used finite differences schemes to evolve the lattice-Boltzmann transport equation in generalized coordinates [23,24], but they demand a new discretization scheme for each new coordinate system and different implementations of the boundary conditions in different directions. Further approaches were more general, allowing to simulate the NSE on any axisymmetric coordinate system [25,26]. In 2012 a more general approach in two dimensions was developed by Budinsky for both the shallow water and the Navier–Stokes equations [27]. In that scheme, the equations are written in general coordinates, and the additional geometric terms (containing the Jacobian and Jacobian spatial derivatives) are introduced as forcing terms in the collision operator. The main advantage of this model is that all information about the curvilinear coordinates is included in the equilibrium distribution function and the forcing term, and no further discretization or interpolation steps are needed. Simultaneously with the Budinsky proposal, M. Mendoza [1] introduced in his doctoral thesis a new strategy to built lattice-Boltzmann models for fluids on any three-dimensional curvilinear coordinate system. The strategy also reproduces in the macroscopic limit the desired equations in generalized coordinates, but using the metric tensor and Christoffel symbols instead of the Jacobian. In contrast with Budinsky's method, the forcing terms are included both in the equilibrium functions and in the macroscopic quantities, following a procedure similar to that by Guo et al. [28] and, reaching second-order accuracy. Again, the strategy does not require neither a specific discretization scheme for each problem nor additional interpolation steps. This model has been successfully used to study the Dean's instability in ellipsoidal coordinates [29], the flow through randomly curved media [30] and, more recently, the energy dissipation due to curvature [31]. The Budinsky and Mendoza models are built to simulate the NSE in curvilinear coordinates, and an equivalent model to simulate the wave equation has not been constructed.

In this work propose a different model to the one by Chopard et al. [18,32] to simulate scalar acoustic waves, and we extend it on generalized coordinates by following a similar procedure to the one proposed by M. Mendoza [1], but with very different forcing terms and reproducing a different macroscopic equation: the scalar wave equation instead of the NSE. The proposed model is completely general, and can be used to reproduce the three-dimensional wave equation in any coordinate system without any interpolation scheme, just by introducing the corresponding metric tensor and Christoffel symbols in the general expressions for the forcing terms and the macroscopic quantities. Moreover, the method reaches stability and second-order accuracy by employing just a simple D3Q7 velocity scheme, constituting a reliable alternative for the simulation of acoustic waves in complex geometries. Section 2 reviews the LBM proposed by Chopard et al., derives an alternative form by using Hermite polynomials and compares their performances in the simple case of a point source in two dimensions. Section 3 extends the alternative form to generalized coordinates, including the general recipe to build the LBM for waves on any coordinate system. The model is tested in Section 4 by simulating the acoustic waves inside a cylinder, a trumpet and a torus, finding second-order accuracy. Finally, Section 5 summarizes the main results and conclusions. Videos of the simulations can be found in the supplementary material attached to this manuscript.

2. LBM for waves in Cartesian coordinates

Let us start from the lattice-Boltzmann's equation with the Bhratnagar–Gross–Krook approximation [33],

$$f_i\left(\vec{x}+\vec{\xi}_i\delta_t,\,\vec{\xi}_i,\,t+\delta_t\right)-f_i\left(\vec{x},\,\vec{\xi}_i,\,t\right)= -\frac{\delta_t}{\tau}\left[f_i\left(\vec{x},\,\vec{\xi}_i,\,t\right)-f_i^{eq}\left(\vec{x},\,\vec{\xi}_i,\,t\right)\right], \tag{1}$$