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On the properties of *k*-balancing numbers

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Abstract In this study, a generalization of the sequence of balancing numbers called as k-balancing numbers is considered and some of their properties are established. Further, the balancing polynomials that are the natural extension of k-balancing numbers are presented and observe that many of their properties admit straightforward proofs. The derivatives of these polynomials in the form of convolution are also discussed.

1. Introduction

The concept of balancing numbers came into existence after an article [1] by Behera and Panda wherein, they defined a balancing number n as solution of the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$, calling r as the balancer corresponding to n. First few balancing numbers are $1, 6, 35, 204$ and 1189 with balancers $0, 2, 14, 84$ and 492 respectively. The sequence of balancing numbers has been studied extensively and generalized in many ways [2–7,9–11,13,16]. In [8], Liptai et al. generalized the concept of balancing numbers in the following way. For y, k, l be fixed positive integers with $y \geqslant 4$, a positive integer x with $x \leqslant y-2$ is called a (k,l)-power numerical center for y if $1^k+\cdots+(x-1)^k=(x+1)^l+\cdots+(y-1)^l$. Several effective and ineffective finiteness results were also proved for (k,l)-power numerical centers in [8]. In [16], Szakáács has studied a further generalization of balancing numbers namely multiplying balancing numbers defined in the following way: A positive integer n is called a multiplying balancing number if $1\cdot 2,\ldots,(n-1)=(n+1)(n+2),\ldots,(n+r)$, for some positive integer r which is called as multiplying balancer corresponding to the multiplying balancing number n. He proved that the only multiplying balancing number is $n=7$ with the multiplying balancer $r=3$. As a generalization of the notion of a balancing number in [3], Bérczes et al. called a binary recurrence $R=R(A,B,R_0,R_1)$, a balancing sequence if $R_1+R_2+\ldots+R_{n-1}=R_{n+1}+R_{n+2}+\ldots+R_{n+k}$, holds for some $k \geqslant 1$ and $n \geqslant 2$.

The present article is organized as follows. In Section 2, a generalization of sequence of balancing numbers which we call as k-balancing sequence $\{B_{k,n}\}_{n=0}^{\infty}$ depending on one real parameter k, is considered and some of their properties are investigated. In Section 3, the balancing polynomials that are the natural extension of k-balancing numbers are introduced and many of their properties are established. The derivatives of these polynomials in the form of convolution of balancing polynomials are presented in Section 4. As an application of balancing polynomials, a balancing based coding method is also developed in the final section.

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2. k-Balancing numbers and their properties

Polynomials and balancing numbers are well related. In [14], Ray has applied Chebyshev polynomials in factorization of balancing and Lucas-balancing numbers. In [10], Özkoc has introduced k-balancing numbers and presented some relations in terms of these numbers. These relations generalized some well known results concerning the relation between the determinant and Chebyshev polynomials, which is due to the $B_n(k)$.

The following definition of k-balancing numbers is given in [10].

Definition 2.1. For any positive number k, the k-balancing numbers, denoted by $\{B_{k,n}\}_{n=0}^{\infty}$ defined recursively by

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1} \quad n \geq 1, \tag{1}$$

with the initials $B_{k,0} = 0$ and $B_{k,1} = 1$.

First few k-balancing numbers are

$$B_{k,0} = 0, \quad B_{k,1} = 1, \quad B_{k,2} = 6k, \quad B_{k,3} = 36k^2 - 1$$

$$B_{k,4} = 216k^3 - 12k, \quad B_{k,5} = 1296k^4 - 108k^2 + 1 \text{ and so on.}$$

Notice that, $k = 1$ in (1) gives the sequence of balancing numbers. Also observe that, (1) is a second order difference equation with auxiliary equation $\alpha^2 = 6k\alpha - 1$, whose roots are $\alpha_1 = 3k + \sqrt{9k^2 - 1}, \alpha_2 = 3k - \sqrt{9k^2 - 1}$. Clearly,

$$\alpha_1 + \alpha_2 = 6k, \quad \alpha_1\alpha_2 = 1, \quad \alpha_1 - \alpha_2 = 2\sqrt{9k^2 - 1}. \tag{2}$$

The following results are some important identities for k-balancing numbers.

Lemma 2.2. *For any integer* $n \geqslant 1, \alpha_1^{n+2} = 6k\alpha_1^{n+1} - \alpha_1^n$ *and* $\alpha_2^{n+2} = 6k\alpha_2^{n+1} - \alpha_2^n$.

Proof. As α_1 and α_2 are roots of the equation $\alpha^2 = 6k\alpha - 1$, $\alpha_1^2 = 6k\alpha_1 - 1$ and $\alpha_2^2 = 6k\alpha_2 - 1$. The desired results are obtained by multiplying α_1^n and α_2^n to both the equations respectively. □

Lemma 2.3 (*Binet's formula*). *The closed form nth k-balancing number is* $B_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}$, *where* $\alpha_1 = 3k + \sqrt{9k^2 - 1}, \alpha_2 = 3k - \sqrt{9k^2 - 1}$.

Proof. By method of induction, clearly the result is true for $n = 0$ and $n = 1$. Assume that it is true for all i such that $0 \leqslant i \leqslant m + 1$ for some positive integer m. Now by (1), we obtain

$$\begin{aligned} B_{k,m+2} &= 6kB_{k,m+1} - B_{k,m} = 6k\frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} - \frac{\alpha_1^m - \alpha_2^m}{\alpha_1 - \alpha_2} \\ &= \frac{\alpha_1^m[6k\alpha_1 - 1] - \alpha_2^m[6k\alpha_2 - 1]}{\alpha_1 - \alpha_2} = \frac{\alpha_1^{m+2} - \alpha_2^{m+2}}{\alpha_1 - \alpha_2}, \end{aligned}$$

which ends the proof. □

Lemma 2.4. *Let* $\binom{n}{i}$ *denote the usual notation for combination. Then for any integer* $n \geqslant 0$, $\sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}6^ik^iB_{k,i} = B_{k,2n}$.

Proof. By virtue of the Binet's formula,

$$\begin{aligned} \sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}6^ik^iB_{k,i} &= \sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}6^ik^i\left[\frac{\alpha_1^i - \alpha_2^i}{\alpha_1 - \alpha_2}\right] \\ &= \frac{1}{\alpha_1 - \alpha_2}\left[\sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}(6k\alpha_1)^i - \sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}(6k\alpha_2)^i\right] \\ &= \frac{1}{\alpha_1 - \alpha_2}[(6kr_1 - 1)^n - (6kr_2 - 1)^n] = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{\alpha_1 - \alpha_2} = B_{k,2n}, \end{aligned}$$

which completes the proof. □

The proof of the following results is omitted as they can be easily shown by Binet's formula.

Proposition 2.5. *For* $n \geqslant 1, B_{k,-n} = -B_{k,n}$.

Proposition 2.6. $\sum_{i=0}^{n}B_{k,i+j} = -\frac{1}{18k}\left[3(B_{k,n+j}B_{k,n+j+1}) + B_{k,j} + B_{k,j-1}\right]$.

Lemma 2.7. *For natural numbers* p, q, r, *the following identities are valid.*

(*a*) $B_{k,p+q-1} = B_{k,p}B_{k,q} - B_{k,p-1}B_{k,q-1}$
(*b*) $B_{k,p+q-2} = \frac{1}{6k}\left[B_{k,p}B_{k,q} - B_{k,p-2}B_{k,q-2}\right]$
(*c*) $B_{k,p+q+r-3} = \frac{1}{6k}\left[B_{k,p}B_{k,q}B_{k,r} - 6kB_{k,p-1}B_{k,q-1}B_{k,r-1} + B_{k,p-2}B_{k,q-2}B_{k,r-2}\right]$

Proof. Proof of (a) is easy by Binet's formula. Now, inserting (1) in (a),

$$\begin{aligned} B_{k,p+q-2} &= B_{k,p}B_{k,q-1} - B_{k,p-1}B_{k,q-2} \\ &= B_{k,p}\left[\frac{B_{k,q} + B_{k,q-2}}{6k}\right] - B_{k,q-2}\left[\frac{B_{k,p} + B_{k,p-2}}{6k}\right] \\ &= \frac{1}{6k}\left[B_{k,p}B_{k,q} - B_{k,p-2}B_{k,q-2}\right], \end{aligned}$$

which completes the proof of (b). Proof of (c) is analogous to (b). □

Lemma 2.8. *For any integer* $n \geqslant 1, \alpha_1^n = \alpha_1B_{k,n} - B_{k,n-1}, \alpha_2^n = \alpha_2B_{k,n} - B_{k,n-1}$.

Proof. Using (2), we get

$$\alpha_1^2 = 6k\alpha_1 - 1 = \alpha_1B_{k,2} - B_{k,1},$$

$$\alpha_1^3 = 6k\alpha_1^2 - \alpha_1 = (36k^2 - 1)\alpha_1 - 6k = \alpha_1B_{k,3} - B_{k,2}.$$

Consequently, $\alpha_1^n = \alpha_1B_{k,n} - B_{k,n-1}$. The second identity can be proved analogously. □