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Revisiting fractional Gaussian noise

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HIGHLIGHTS

- The article gives the expressions of the complete form of the fractional Gaussian noise and the fractional Brownian motion of the Weyl type (fGn).
- The article proposes the analytic expressions of the impulse response function and the frequency transfer function of the fractional-order filter that produces the fGn under the excitation of white noise.
- The article suggests that the statistical dependences of fGn may be described from the point of view of the frequency transfer function or the impulse response function of the fractional-order filter.

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ABSTRACT

The contributions given in this article are mainly in two folds with regard to the fractional Gaussian noise of the Weyl type. The first one is to present a complete form of fGn, which is taken as the output of a fractional-order filter driven by white noise. The second is to propose the analytic expressions of the impulse response function and the frequency transfer function of that fractional-order filter. We also give an expression of the fractional Brownian motion of the Weyl type as a consequence of the present fGn. Besides, we suggest that the statistical dependences of fGn may be described based on the frequency transfer function or the impulse response function of the fractional-order filter.

1. Introduction

Time series with long-range dependence (LRD), observed by Hurst in 1950 (Hurst [1]), attracts the interests of researchers in various fields ranging from physics to computer science, see e.g., Beran [2], Levy-Vehel [3], Peng et al. [4], Buldyrev et al. [5], Gneiting [6], Muniandy et al. [7], Gu et al. [8], Pinchas [9], Borgnat et al. [10], Chen et al. [11], Cattani et al. [12], Goychuk and Hänggi [13], simply citing a few. Among the various models of time series with LRD, a widely used one is the fractional Gaussian noise of the Weyl type (fGn in short) introduced by Mandelbrot and van Ness [14].

Let $B(t)$, $t \in (0, \infty)$, be the standard Brownian motion (Bm). Then, $B'(t)$ has to be considered in the domain of generalized functions over the Schwartz space of test functions, see e.g., Gelfand and Vilenkin [15] or Griffel [16] for generalized functions.

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Let $B_H(t)$ be the fractional Brownian motion of the Weyl type (fBm for short). Then,

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}[(t-u)^{H-0.5} - (-u)^{H-0.5}]dB(u) + \int_{0}^{t}(t-u)^{H-0.5}dB(u)\right\}, \tag{1.1}$$

where $H \in (0, 1)$ is the Hurst parameter. $B_H(t)$ reduces to Bm when $H = 1/2$. It is self-similar in the sense that

$$B_H(at) \equiv a^H B_H(t), \quad a > 0, \tag{1.2}$$

where $\equiv$ denotes the equality in probability distribution.

The autocorrelation function (ACF) of $B_H(t)$ is expressed by

$$r_{B_H}(t, s) = \frac{V_H}{(H+1/2)\Gamma(H+1/2)}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \tag{1.3}$$

where V_H is the strength of $B_H(t)$ in the form

$$V_H = \mathrm{Var}[B_H(1)] = \Gamma(1-2H)\frac{\cos \pi H}{\pi H}. \tag{1.4}$$

The power spectrum density (PSD) of $B_H(t)$ is given by (Flandrin [17])

$$S_{B_H}(t, \omega) \sim \frac{1}{|\omega|^{2H+1}}(1 - 2^{1-2H}\cos 2\omega t). \tag{1.5}$$

Both (1.3) and (1.5) imply that $B_H(t)$ is nonstationary as they are time varying. In addition, (1.5) exhibits that $B_H(t)$ is a type of $1/f$ noise. Accordingly, it is of LRD for $H \in (0, 1)$.

Denote by $G(t)$ the increment process of $B_H(t)$. It is given by

$$G(t) = B_H(t+a) - B_H(t), \ \ a > 0. \tag{1.6}$$

The above is the fGn with the ACF in the form

$$C_G(\tau) = \frac{V_H \varepsilon^{2H}}{2}\left[\left(\left|\frac{\tau}{\varepsilon}\right| + 1\right)^{2H} + \left|\left|\frac{\tau}{\varepsilon}\right| - 1\right|^{2H} - 2\left|\frac{\tau}{\varepsilon}\right|^{2H}\right], \tag{1.7}$$

where $\varepsilon > 0$ is the parameter utilized for smoothing $B_H(t)$ so that the smoothed one is differentiable [[14], p. 427–428].

There are three subclasses of the fGn. When $H \in (0.5, 1)$, $C_G(\tau)$ is positive and finite for all τ. It is non-integrable and the corresponding series is of LRD. If $H \in (0, 0.5)$, the integral of $C_G(\tau)$ is zero and $C_G(0)$ diverges when $\varepsilon \to 0$. Hence, it is of short-range dependence (SRD) when $H \in (0, 0.5)$. Moreover, $C_G(\tau)$ changes its sign and becomes negative for some τ proportional to ε in this parameter domain [[14], p. 434]. It reduces to white noise when $H = 0.5$.

The PSD of fGn is given by (Li and Lim [18])

$$S_G(\omega) = V_H \sin(H\pi)\Gamma(2H+1)\,|\omega|^{1-2H}. \tag{1.8}$$

Both (1.7) and (1.8) mean that the fGn is stationary because they are time invarying.

One may use (1.8) to describe the statistical dependences of fGn in frequency domain. As a matter of fact,

$$S_G(0) = \int_{-\infty}^{\infty} C_G(\tau)d\tau \to \infty \ \text{ if } \ H \in (0.5, 1), \tag{1.9}$$

which implies that the fGn is a kind of $1/f$ noise for $H \in (0.5, 1)$ and reflects its LRD property in frequency domain. On the other hand, when $H \in (0, 0.5)$, we have

$$S_G(0) = \int_{-\infty}^{\infty} C_G(\tau)d\tau < \infty \ \text{ for } \ H \in (0, 0.5), \tag{1.10}$$

which represents its SRD from a view of spectrum.

Denote by $w(t)$ the normalized white noise with the PSD $S_w(\omega)$ in the form

$$S_w(\omega) = 1. \tag{1.11}$$

The ACF of $w(t)$ is given by

$$R_w(\tau) = \delta(\tau). \tag{1.12}$$

where $\delta(\cdot)$ is the Dirac-Delta function. Following Li [19], Li and Chi [20], Makse et al. [21], we have

$$G(t) = w(t) * h(t), \tag{1.13}$$

where $*$ stands for the convolution operation and $h(t)$ is the impulse response function of a fractional-order filter in the form

$$h(t) = \mathrm{F}^{-1}\left\{\sqrt{\mathrm{F}\left[C_G(t)\right]}\right\}, \tag{1.14}$$