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Bounding the inefficiency of atomic splittable selfish traffic equilibria with elastic demands

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ABSTRACT

We determine the exact upper bound of the inefficiency of atomic splittable selfish traffic equilibria with elastic travel demand with and without road pricing. In the previous results, only pseudo-approximation bound were obtained for this case. By comparison, we also conclude that the traffic equilibrium with elastic demand may be worse than the corresponding fixed demand case, which implying that the demands' elastic can have a negative effect on the quality of equilibrium solutions. Finally, we propose a road pricing mechanism. We prove that there are optimal tolls in general network, atomic players and elastic travel demand setting.

1. Introduction

Logistic and freight companies routinely transport goods between different points in a transportation network to serve their clients. They make use of trucks, trains or ships to carry goods from source to destination points. These vehicles have to traverse parts of a network that is also shared by other competitors and civil traffic. Companies that provide this service compete in at least two dimensions: the price they charge for shipping and the service level they provide. To improve their competitive advantage, these companies need to be strategic in how they deliver the goods and minimize costs and delivery times.

Therefore, one fundamental problem arising in the management of large-scale traffic networks is the following: given the demands of traffic between each pair of nodes in a network, find an assignment of traffic to paths so that the sum of all travel times (the total cost) is minimized. However, in the absence of network regulation, network users are free to act according to their own interests, without regard to overall network performance. In these settings, it is often difficult or even impossible to impose optimal or near-optimal routing strategies on the traffic in a network.

W will study this competition from the perspective of *noncooperative game theory*, and use the Nash equilibrium as the solution concept of the game. In a traffic network games, the large logistic firms, bus companies, or freight shipping centers are referred to *players*, and the trucks, trains or ships are referred to *users*. The congestion on a link is modeled by a nondecreasing cost function. Such functions map the total flow on a link to the time needed to traverse this link. The cost on a link is defined as time on that link. The players route demand with minimum cost. The solution reached by players selfishly routing their flow is called the Nash equilibrium or Wardropian User Equilibrium (Wardrop, 1952), under which no competitor has any regret after seeing what all competitors have done. The total social cost of routing flow is defined by the sum over all link costs. The minimized total social cost is called social optimum. It is well known that a Nash equilibrium can be

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inefficient with respect to a social optimum (Pigou, 1920; Rapoport and Chammah, 1965). Namely, there exists an efficiency loss of the user equilibrium compared to a system optimum (Braess, 1968; Dubey, 1986).

The main goal of this paper is to quantify the efficiency loss of atomic splittable selfish routing game with elastic demands, i.e., the upper bound of *price of anarchy* (PoA) (Koutsoupias and Papadimitriou, 1999), and investigate the difference on this issue between elastic demands and fixed demands. Finally, we study the efficiency gain through road pricing mechanism.

The traffic network game belongs to *Congestion game* (Rosenthal, 1973), which includes the following three situations:*nonatomic*, *atomic unsplittable*and *atomic splittable*. In this work, we focus on the third case–atomic splittable selfish routing game, where some players may control a significant part of the entire demand. The users belonging to the same player are fully cooperative while different players are fully competitive.

1.1. Our results

As far as we know, in the context of atomic splittable selfish traffic equilibria with elastic demands, none had succeeded in obtaining an exact upper bound of PoA. For this reason, this article presents the first exact upper bounds of efficiency loss of this case. In addition, we develop a road pricing mechanism, which is able to reduce the upper bounds. Namely, the Nash equilibrium (or User Equilibrium) is closer to the social optimum under the optimal pricing scheme. Our Contribution in this setting is as follows.

(1) We firstly determine the exact upper bound of the inefficiency of atomic splittable selfish traffic equilibria with elastic travel demand. We show that the worst-case PoA is exactly 1.667 when link cost functions and demand functions are restricted to affine; and the worst-case PoA is exactly $1+\frac{3p(\sqrt{p}+1)^{p}(p-1)}{2^{p+1}(p^{\frac{3}{2}}-1)}$ if link cost functions are restricted to polynomials and demand functions to negative exponentials, where p is the polynomials' degree. Therefore, for affine, quadratic, cubic and BPR (Bureau of Public) polynomials link cost functions cases, the upper bounds of PoA are approximately 3.391, 6.467, 14.018 respectively.

(2) We firstly compare the results of elastic demand case with fixed case, and conclude that the worst-case PoA for elastic demand case are larger than that of minimum upper bound presented by Harks (2011) or Roughgarden et al. (2011) for fixed demand case. That is the traffic equilibrium with elastic demand may be worse than the corresponding fixed demand case, which implying that the demands' elastic can have a negative effect on the quality of equilibrium solutions.

(3) We develop a road pricing mechanism which is able to reduce the efficiency loss of Nash equilibrium, and firstly prove that there are optimal tolls in general network, atomic players and elastic travel demand setting. We also derive that, under the optimal toll scheme, the worst-case PoA for elastic demands case is able to reduce to 3/2 which is the optimum level of fixed demand case.

1.2. Related literatures

A first result for exactly quantifying the PoA was given by Koutsoupias and Papadimitriou (1999) in the context of a load balancing game in communication networks. The pioneering study of the PoA in *nonatomic* network games is due to Roughgarden and Tardos (2002), after which the idea of bounding the PoA in *nonatomic* network games has become well-studied (Chau and Sim, 2003; Cominetti et al., 2009; Correa et al., 2004; Correa et al., 2005; Correa et al., 2007; Correa et al., 2008; Harks, 2011; Hayrapetyan et al., 2006; Roughgarden, 2003; Roughgarden and Tardos, 2004). The readers may refer to Roughgarden (2005a,b) for a recent comprehensive review of this research subject.

For the *atomic unsplittable* network games, introduced by Rosenthal (1973), where each player must route its traffic on a single path or flow of each player is unsplittable, Roughgarden and Tardos (2002), Awerbuch et al. (2005), Christodoulou and Koutsoupias (2005) studied the PoA in the unsplittable variant for linear link cost functions. Aland et al. (2006) then proved exact bounds on the PoA for general polynomial link cost functions in this case.

Orda et al. (1993), who noted that existence of equilibria follows directly from the classical result about concave games of Rosen (1965), considered the *atomic splittable* case for the first time. Subsequently, Roughgarden (2005a,b) studied the PoA for atomic selfish routing games; he proved that all known bounds on the PoA of selfish routing continue to hold for networks with a finite set of atomic players who can split their flow fractionally. And more recently, Cominetti et al. (2009) studied the atomic splittable selfish routing model in which the flow of every commodity forms a coalition. Using a variational inequality approach, they presented bounds of 1.5, 2.56, and 7.83 on the PoA, for linear and polynomial cost functions of degree two and three, respectively. As noted by Cominetti et al., these positive bounds directly carry over to the case of nonatomic network congestion games with arbitrary coalitions. However for polynomials of larger degree, their approach does not yield bounds. Other results for studying the atomic splittable network congestion games can be found e.g., in (Altman et al., 2002; Bhaskar et al., 2009; Cominetti et al., 2006; Harks, 2011; Hayrapetyan et al., 2006; Roughgarden et al., 2011; Yang et al., 2008).

All the results mentioned above have overwhelmingly considered the fixed demand case. Although Chau and Sim (2003) and Yang et al. (2010) examined the PoA with elastic demand, they studied the nonatomic situation. Furthermore, they only proposed a "weaker" bound of PoA. The bound is "weaker" in the sense that, unlike its counterpart with fixed demand, the