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Single thermal zone balance solved by Transfer Function Method

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Abstract

We present an algorithm that uses the *Z*-transform operator to face the problem of heat transmission in a single thermal zone composed by multilayered walls. The method is very flexible and could be adopted to calculate the transfer function coefficients able to simulate the thermal behaviour of a room in free floating. Knowing the transfer function coefficients, it is possible to simulate the dynamic profile of each inner surfaces temperature and furthermore of the inner air temperature.

The proposed algorithm is fully described granting maximum clarity. The explicitness of all steps of the calculus make possible the definition of a method that is able to vary all of the calculus parameters such as sampling *g* period, number of roots, number of poles or number of coefficients.

To assess the reliability of the algorithm, we carried out a comparison between simulation data obtained from our method, from Fourier steady-state algorithm and those obtained from TRNSYS.

Keywords: Transfer Function Method; Energy simulation in buildings; Poles; Roots; Sampling period

1. Introduction

The "ASHRAE" procedure called Transfer Function Method (TFM), largely diffused in designing and simulating HVAC, analyses the heat transmission problems using *Z*-transform. *Z*-transform is a suitable mathematical operator when simulation models have to deal with data inputs or outputs discrete in the time domain such as climatic data [1,2].

As well as the other mathematical procedures related to the study of dynamic thermal behaviour of buildings, TFM is an approximated mathematical approach because the exact solution would require an infinite number of calculus [3].

Excluding approximations linked to physical assumptions, the most important source of inaccuracy is due to the truncation of the infinite coefficients that constitute the Transfer Functions (TFs) [4,5]. When TFs are in the form of $\frac{\mathrm{num}(z)}{\mathrm{den}(z)}$, the absolute value of the coefficients decreases very quickly increasing the order of the addendum. This particular feature, and the fact that in the employed equations, the coefficients are in an explicit form, make easier the truncation procedure of TFs.

To execute a correct truncation procedure, it is necessary to evaluate the effect in the numerical response linked to the insertion or the elimination of a coefficient. This evaluation can be obtained only having a large number of coefficients.

In the ASHRAE manuals, available data permit to carry out this evaluation only for the TFs of walls. On the contrary, TFs of cooling load are described by expressions like:

$$K(z) = \frac{v_0 + v_1 z^{-1}}{1 + w_1 z^{-1}} \tag{1}$$

which employs no more than three coefficients globally.

In fact, while in the calculus of the TFs for walls was used a rigorous mathematical approach, in the case of cooling load of a thermal zone was adopted a numerical procedure able to obtain not the TFs but only the time response of the system solicited by an unitary impulse signal. This response in the time domain, describing the nature of the system, is characterised by a slower time decay in presence of higher values of thermal inertia. Practically, to employ this function, we should select a large number of coefficients.

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Furthermore, in this case, the absolute value of the coefficients do not decrease quickly increasing their order.

When TFs are in the form of equation (1), the denominator contains terms linked exclusively to the system. In other words, $D(z)$ "contains" the system, to limit the number of coefficients make more approximate the description of the system. Not having the function $D(z)$, in the "ASHRAE" method, they obtained a denominator by using a procedure founded on the hypothesis that the response of the system contains an infinite series of exponential terms with a negative time constant. [6,7] Then, the procedure should be able to identify first dominant poles of TFs and making possible to write the denominator in the form:

$$D(z) = (1 - CR_1 z^{-1})(1 - CR_2 z^{-1})(1 - CR_3 z^{-1}) \tag{2}$$

In case of thermal zone with low value of thermal inertia, time decay of the response is so fast to prevent the calculus of CR_2 and CR_3. This means that thermal zones delimited by walls with low thermal inertia, described by TFs with at least three or four poles, have a thermal behaviour that can be described employing an unique pole relate to CR_1.

Clearly, in the above procedure, there is not a fully strict approach from the mathematical point of view. It depends from the employed method to solve the system's equations.

To solve these problems, authors present a method to solve in a strict mathematical approach a system of linear differential equations representing, under some physical hypothesis, the thermal dynamic behaviour of a thermal zone. This procedure determines the TFs of the system by using Z-transform, finding all the coefficients that we require, independently from their order, thermal inertia of the room and walls.

2. Thermal balance of the thermal zone

On the external surface, the thermal balance is simplified by using the concept of air–sol temperature. The inner surface presents a heat flux due to the convective exchange with the internal air and the radiative exchange with other surfaces. It also receives a thermal flux from the internal environment due to electric devices or cooling plants. The thermal exchange between the glass and the internal air is summarized by an exchange coefficient. Solar radiation is distributed by means of a simple linear model related to the extension of the inner surfaces.

In order to evaluate the accuracy of calculations carried out by using a given set of ZT coefficients, we have to compare them with a reference response coming from a procedure having a different mathematical background and even able to give the time continuous response of the system.

3. Solving the equation's system

If a system is solicited by an input signal $i(t)$ time variable, that produces an output signal $o(t)$, in the Z-transform domain, the link we have to determine is $G(z) = \frac{O(z)}{I(z)}$, in which $I(z)$ and $O(z)$ are the Z-transforms (ZT) of $i(t)$ and $o(t)$.

The calculus can be done using:

$I(s) = \mathrm{LT}[i(t)]$, $O(s) = \mathrm{LT}[o(t)]$ and the transfer function of the system in the Laplace domain $G(s)$.

In this case, the transfer function in the Z domain is:

$$G(z) = \frac{O(z)}{I(z)} = \frac{\mathrm{ZT}[O(s)]}{\mathrm{ZT}[I(s)]} = \frac{\mathrm{ZT}[I(s)G(s)]}{\mathrm{ZT}[I(s)]} = \frac{\mathrm{num}(z)}{\mathrm{den}(z)} \tag{3}$$

where $\mathrm{num}(z)$ is a polynomial called *numerator* and $\mathrm{den}(z)$ is a polynomial called *denominator*. Assuming a linear ramp as input signal, applying the Heaviside theorem, we obtain:

$$O(s) = \frac{1}{s^2}G(s) = \frac{1}{s^2}\frac{\mathrm{NUM}(s)}{\mathrm{DEN}(s)} = \frac{C_0}{s^2} + \frac{C_1}{s} + \sum \frac{\mathrm{res}_n}{(s - s_n)} \tag{4}$$

where s_n are s such that $\mathrm{DEN}(s_n) = 0$ also called poles of the system; $\mathrm{res}_n = \left.\frac{\mathrm{NUM}(s)}{s^2 \mathrm{DEN}'(s)}\right|_{s=s_n}$ are the residuals linked to the poles of the system; C_0 and C_1 are the residuals linked to the double pole in the origin of axis due to the linear ramp input;

$$C_0 = \left[\frac{\mathrm{NUM}(s)}{\mathrm{DEN}(s)}\right]_{s=0} C_1 = \left[\frac{\mathrm{NUM}'(s)\mathrm{DEN}(s) - \mathrm{NUM}(s)\mathrm{DEN}'(s)}{\mathrm{DEN}(s)^2}\right]_{s=0};$$

$$\mathrm{DEN}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{DEN}(s); \qquad \mathrm{NUM}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{NUM}(s);$$

Note that $\mathrm{num}(z) \neq \mathrm{ZT}[\mathrm{NUM}(s)]$, $\mathrm{den}(z) \neq \mathrm{ZT}[\mathrm{DEN}(s)]$.

In the Z-transform domain, we have:

$$G(z) = \frac{O(z)}{I(z)} = \frac{\frac{C_0 \Delta}{(1-z^{-1})^2} + \frac{C_1}{1-z^{-1}} + \sum \frac{\mathrm{res}_n}{1-\mathrm{e}^{s_n \Delta} z^{-1}}}{\frac{C_0 \Delta}{z(1-z^{-1})}} = \frac{\mathrm{num}(z)}{\mathrm{den}(z)} \tag{5}$$

where Δ is the sampling period.

$$\mathrm{den}(z) = \prod (1 - \mathrm{e}^{-s_n \Delta} z^{-1}) \tag{6}$$

where the sum and the product are extended to all the poles of the system.

Assuming the one-dimensional heat transfer in a homogeneous and isotropic layer with constant thickness L, letting be $\theta(x, t)$ the temperature and $q(x, t)$ the heat flux at the time t along x direction, the system equations that represent the thermal balance of the layer can be represented in a compact matrix notation:

$$\begin{vmatrix} T_\mathrm{e} \\ Q_\mathrm{e} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} T_\mathrm{i} \\ Q_\mathrm{i} \end{vmatrix} \tag{7}$$

where T_e is the LT of the temperature t_e of the external side of the layer, Q_e the LT of the heat flux q_e on the external side of