Expert Systems with Applications 42 (2015) 6522–6529

Contents lists available at ScienceDirect

Expert Systems with Applications

journal homepage: www.elsevier.com/locate/eswa

The slack-based measure model based on supporting hyperplanes of production possibility set

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ARTICLE INFO

Article history:
Available online 17 April 2015

Keywords:
Data Envelopment Analysis (DEA)
Slack based measure (SBM) model
Hyperplane
Production possibility set (PPS)

ABSTRACT

The original slack-based measure (SBM) model evaluates the efficiency of decision making units (DMUs) referring to the furthest frontier point within a range. Hence the projection may go to a remote point on the efficient frontier which may be inappropriate as a reference point. In this paper we propose a new variant for the improvement of efficiency scores in SBM models. It is based on the determination of strong hyperplanes of the production possibility set (PPS). The approach presented here improves the currently used Tone method both, from the time consumption and computational points of view. Comparative examples, as well as a case study, are given to illustrate the new procedure.

1. Introduction

Data Envelopment Analysis (DEA) is a relatively new "data oriented" approach for the evaluation of the performance of a set of entities called Decision Making Units (DMUs), which transform multiple inputs into multiple outputs. The DEA research started by publication of the essential paper (Charnes, Cooper, & Rhodes, 1978). The definition of DMUs is very generic and flexible. Recent years have seen a great variety of applications of DEA models for performance and efficiency evaluation of many different kinds of entities engaged in many different activities and contexts, in many different countries (Akçay, Ertek, & Büyüközkan, 2012;Bayraktar, Tatoglu, Turkyilmaz, Delen, & Zaim, 2012;Charles & Zegarra, 2014;Cooper, Seiford, & Tone, 1999;Emrouznejad, Parker, & Tavares, 2008; Rezaei, Ortt, & Scholten, 2012;Wanke & Barros, 2014).

As it is known in DEA, the observed DMUs define the production possibility set (PPS). The PPS is a polyhedral convex set whose vertices correspond to the efficient DMUs. The DEA models find the projection of the inefficient DMUs on the efficient frontier of the PPS. That is why the hyperplanes of the PPS are helpful and important in the process of efficiency evaluation of DMUs, as well as in sensitivity and stability analysis (Jahanshahloo, Hosseinzadeh Lotfi, Shoja, Sanei, & Tohidi, 2005a;Khanjani Shiraz, Charles, & Jalalzadeh, 2014). Unfortunately, there are only a few papers in existence which deal with hyperplanes of the PPS and their usage. For instance, Jahanshahloo, Hosseinzadeh Lotfi, and Zohrehbandian (2005b) proposed a method for obtaining the efficient frontier using the integer programming model with binary variables. Yu, Wei, Brockett, and Zhou (1996) studied the structural properties of DEA efficient surfaces of the PPS under the generalized DEA model. Jahanshahloo, Hosseinzadeh Lotfi, Zhiani Rezai, and Rezai Balf (2007), Jahanshahloo, Shirzadi, and Mirdehghan (2009) and Jahanshahloo, Hosseinzadeh Lotfi, and Akbarian (2010) developed the algorithms which are used to find defining hyperplanes of the PPS. Amatatsu and Ueda (2012) show an alternative use of the efficient facets in DEA. Specifically, they indicate that once all facets of the DEA technology is identified, decision maker is able to estimate the potential changes in some inputs and outputs, while fixing other inputs and outputs. Aparicio and Pastor (2014) show least distance measures based on Hölder norms satisfy neither weak nor strong monotonicity on the strongly efficient frontier. They study Hölder distance functions and show why strong monotonicity fails. Along this line, they provide a solution for output-oriented models that allows assuring strong monotonicity on the strongly efficient frontier. Aghayi and Gheleyj Beigi (2014) show that the strong (weak) defining hyperplane is supporting and there exists, at least, one affine independent set with $m+s$ elements of extreme efficient DMUs (extreme efficient and weak efficient virtual DMUs) where m and s are the number of inputs and outputs, respectively. Nasrabadi, Dehnokhalaj, and Soleimani-damaneh (2014) characterize a subset of the production possibility set consisting of production points whose radial projection points lie on the same supporting hyperplane of the PPS. To this end, they consider the CCR and BCC models and establish some theoretical results by utilizing linear programming-based techniques. Determining such a subset of the PPS provides a means to perform

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http://dx.doi.org/10.1016/j.eswa.2015.03.032
0957-4174/

sensitivity analysis of inefficient units. Aparicio and Pastor (2013) show that the Russell output measure of technical efficiency based on closest targets is strongly monotonic when dealing with a full-dimensional strong efficient frontier. This is achieved by replacing non-efficient faces, in the Pareto sense, by linear combinations from the full-dimensional part of the efficient frontier, i.e., extending the efficient facets of the original DEA technology.

One of the main tasks in the DEA models is to find the efficiency score of all (efficient and inefficient) DMUs. Generally speaking, calculation of the efficiency score of the DMUs is investigated in terms of the following aspects. The first aspect is the computational time. It is important to get the results for all DMUs in a short time. The second aspect is computational accuracy; i.e., to calculate the efficiency score of inefficient DMUs with the least possible error. The DEA models cannot satisfy both aspects simultaneously. They just take into account one aspect according to the decision maker's point of view. In most situations, the aim of the decision maker is not just to split the DMUs into inefficient and efficient classes. The aim is, rather, to find ways to improving the efficiency score of inefficient DMUs with the least possible energy, cost or other inputs. It is clear that the calculation of the exact value of the efficiency score of DMUs is very important. The original SBM model evaluates the efficiency of DMUs referring to the furthest frontier point within a given range. Whereas the classical SBM-model projects inefficient DMUs on the efficiency frontier. This may that this projection be a remote point on the frontier. In other words, it is possible that the obtained projection does not located to the closest supporting hyperplane of PPS. This is problematic when aim is removing the inefficiency. That is, when the projected point is remote then removing the inefficiency is difficult.

In an effort to overcome this shortcoming, Tone proposed four variants of the SBM model (Tone, 2010). Tone's method (Tone, 2010) is very interesting, but he deals only with the efficient part (strong hyperplanes) of the PPS, and ignores the inefficient frontiers. On the other hand due to the number and structure of the themes of the SBM model, proposed by Tone, computational time increases considerably. When the efficiency score is computed in all themes for a particular DMU, then the maximum value is selected as the final efficiency score. The second theme consists in finding the facets of the PPS. For this purpose an algorithm was proposed by Tone (for more details, see Tone, 2010). Unfortunately, creating a computer programming code for this algorithm is impossible. That is why the mentioned algorithm can be only used when the number of DMUs is small. In the third theme, the clustering approach was proposed for using the facets' algorithm, but the question is, what type of clustering is better. Besides this, the obtained efficiency scores are local and restricted to the same cluster, and it is not possible to obtain a global efficiency score across the whole PPS. In the fourth theme, a random direction approach is proposed as a way to find facets. Unfortunately, the number of steps in random directions is unknown and this can lead to undesirably long calculations.

In this paper, we first show that the results of Tone's method on the inefficient part of the PPS, are not better than those on the efficient part. Next, we propose a new procedure for finding all facets of the PPS without any clustering or random search. Our procedure is computationally feasible for a large number of DMUs, and as the result, it reduces the massive enumeration of facets.

The rest of this paper is organized as follows. The following section contains the introductory definitions and preliminaries of the paper. In this section, we review and discuss the Tone's method. Main paper's contribution is included in Section 3, where we propose a method for finding strong supporting hyperplanes of the PPS. The proposed procedure, along with three numerical examples, is illustrated in Section 4. Section 5 contains a case study based on a real data set. Its results allow better understanding of the proposed algorithm. Conclusions as the last section summarize the given results, and presents directions for future research in this field.

2. Preliminaries

Suppose we have n DMUs, where all DMUs ($\mathrm{DMU}_j, j = 1, \ldots, n$) have m inputs $\mathbf{x}_{ij}, i = 1, \ldots, m$ and s outputs $\mathbf{y}_{rj}, r = 1, \ldots, s$. We denote the DMU_j by $(\mathbf{x}_j, \mathbf{y}_j), j = 1, \ldots, n$, and the input/output data matrices by $\mathbf{X} = (\mathbf{x}_{ij}) \in \mathbb{R}^{m\times n}$ and $\mathbf{Y} = (\mathbf{y}_{rj}) \in \mathbb{R}^{s\times n}$, respectively, and assume $(\mathbf{X}, \mathbf{Y}) > (\mathbf{0}, \mathbf{0})$. We define the PPS based on constant returns to scale (CRS) as follows:

$$T_c = \left\{ (\mathbf{x}, \mathbf{y}) | \mathbf{x} \geqslant \sum_{j=1}^{n} \lambda_j \mathbf{x}_j, \quad \mathbf{y} \leqslant \sum_{j=1}^{n} \lambda_j \mathbf{y}_j, \quad \lambda_j \geqslant 0, \quad j = 1, \ldots, n \right\}$$

and when we deal with variable returns to scale (VRS) the PPS is defined as:

$$T_v = \left\{ (\mathbf{x}, \mathbf{y}) | \mathbf{x} \geqslant \sum_{j=1}^{n} \lambda_j \mathbf{x}_j,\ \mathbf{y} \leqslant \sum_{j=1}^{n} \lambda_j \mathbf{y}_j,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ j = 1, \ldots, n \right\},$$

in which $\lambda \in \mathbb{R}^n$. We introduce non-negative input and output slacks $\mathbf{S}^- \in \mathbb{R}^m$ and $\mathbf{S}^+ \in \mathbb{R}^s$ to express $\mathbf{x} = \mathbf{X}\lambda + \mathbf{S}^-$ and $\mathbf{y} = \mathbf{Y}\lambda - \mathbf{S}^+$. The SBM model is defined as (Tone, 2001):

$$\begin{aligned} \rho_o = \min & \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \\ s.t. & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \ldots, s \\ & \lambda_j \geqslant 0 \quad j = 1, \ldots, n \\ & s_i^- \geqslant 0 \quad i = 1, \ldots, m \\ & s_r^+ \geqslant 0 \quad r = 1, \ldots, s \end{aligned} \tag{1}$$

Definition 1 (*Reference set*). Let $(\rho_o^*, \lambda^*, S^{-*}, S^{+*})$ be an optimal solution of the model (1). The reference set for DMU $(\mathbf{x}_o, \mathbf{y}_o)$ is defined by $R = \{j | \lambda_j^* > 0, j = 1, \ldots, n\}$.

In this paper we consider the following models

$$\begin{aligned} \max \quad & U\mathbf{y}_o \\ s.t. \quad & V\mathbf{x}_o = 1 \\ & U\mathbf{y}_j - V\mathbf{x}_j \leq 0 \qquad j = 1, \ldots, n \\ & U \geqslant \mathbf{0}, \quad V \geqslant \mathbf{0} \end{aligned} \tag{2}$$

$$\begin{aligned} \max \quad & U\mathbf{y}_o + u_o \\ s.t. \quad & V\mathbf{x}_o = 1 \\ & U\mathbf{y}_j - V\mathbf{x}_j + u_o \leq 0 \qquad j = 1, \ldots, n \\ & U \geqslant \mathbf{0}, \quad V \geqslant \mathbf{0}, \quad u_o \texttt{free} \end{aligned} \tag{3}$$

$$\begin{aligned} \max \quad & U\mathbf{y}_o + u_{o1} - u_{o2} \\ s.t. \quad & V\mathbf{x}_o = 1 \\ & U\mathbf{y}_j - V\mathbf{x}_j + u_{o1} - u_{o2} \leq 0 \qquad j = 1, \ldots, n \\ & U \geqslant \mathbf{0}, \quad V \geqslant \mathbf{0}, \quad u_{o1} \geqslant 0, \quad u_{o2} \geqslant 0 \end{aligned} \tag{4}$$

Definition 2 (*CRS-efficient*). DMU_o is CRS-efficient, if there exists at least one optimal solution (U^*, V^*) for (2), with $(U^*, V^*) > (\mathbf{0}, \mathbf{0})$ such that $U^*\ \mathbf{y}_o$=1, otherwise DMU_o is CRS-inefficient.

Definition 3. (VRS-efficient).DMU_o is VRS-efficient, if there exists at least one optimal solution (U^*, V^*, u_o^*) for (3), with $(U^*, V^*) > (\mathbf{0}, \mathbf{0})$ such that $U^*\ \mathbf{y}_o + u_o^*$=1, otherwise DMU_o is VRS-inefficient.