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A new approach to uncertainty description through accomplishment membership functions

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ABSTRACT

Human linguistic reasoning and statement logic are able to solve uncertain propositions. Similar capabilities are expected to be found on intelligent systems so they are provided with some sort of artificial logic evaluation. Many approaches to uncertainty measurement have been developed before, mainly referring to probability or possibility theories. Some conceptual restrictions are imposed by forcing a distribution function to be conceptually consistent. In this work, conditions imposed to possibility theory are relaxed and the theoretical perspective is oriented to degrees of accomplishment. Conceptual implications and their relation to numerical calculations with respect to a specific class of membership functions are presented. Relation to possibility theory and certainty measurement are discussed to show logical consistency, together with a synthetic numerical example which helps to elaborate conclusions about data dispersion and its relation to the accomplishment proposal.

1. Introduction

The traditional set theory considers a set to be a grouping of objects $S = \{s\}$. Those elements are known to belong to the set by the intuition of containment; however, if the set has a subscribed label, i.e. it represents a concept, the membership of each element to the set can be partial. This was proposed by Zadeh (1965), generalizing set theory so the elements of a set belong to it to a certain extent, depending on how related they are to the concept in question. A membership function (MF) $\mu_A(s) : s \in \Re \rightarrow [0,1]$, assigns a 1 to an element which is completely represented by A's label, while a 0 implies that the element can not be considered as a part of A. So a fuzzy set can be represented as a pair $A = \{s, \mu_A(s)\};\ A \subseteq S$.

Dubois and Prade offered three different ways to understand MFs as pointed by Medasani, Kim, and Krishnapuram (1998): If $\mu_A(x) = 0.8$, (a) 80% of the population declared that x belongs to A (likelihood), (b) 80% of the population described A as an interval which contained x (random set view), and (c) x is at a normalized distance equal to 0.2 from the ideal prototype of A (typicality). Independent to the interpretation of the MF, it is clear that the result of the proposition *"x is A"* can not be true or false, but uncertain. Several techniques to deal with uncertainty through MFs have been developed. An early survey by Medasani et al. (1998) reveals that there is not a specific way to face uncertainty representation and that MFs can be variously defined. They also categorized these approaches depending on their underlying principle as: polling, typicality, heuristic, probability-related, histogram frequency analysis, artificial neural networks, clustering, and mixture decomposition.

Fuzzy logic has provided a way to express uncertainty by capturing the vagueness of linguistic, qualitative, incomplete, or noisy information. Traditionally, uncertainty has been addressed by probability theory as imprecision in variable representation is considered to be statistical in nature. However, the linguistic approximation made by Zadeh allows this vagueness to be addressed in the spirit of its meaning (Zadeh, 1978). By adding this distinction, Zadeh attached a *possibility* interpretation to membership degrees. Although, as said before, there is no specific way to interpret MFs, they are commonly considered as possibility distributions.

Possibility theory has been comprehensively surveyed by Dubois (2006). In the eyes of this author, possibility can be objective when it models a physical property in nature, or epistemic if the uncertainty is derived from the state of knowledge of an agent. There are four ways to understand possibility: Feasibility (ease of achievement), plausibility (propensity), in a logical manner (it is compliant to some information), and deontic (permitted by the law). From these perspectives, the plausibility is the most commonly adopted in research works as it directly shows how sure we are about a certain proposition. Plausibility is also dual related

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to certainty as the later reflects a lack of plausibility of an opposite proposition. However, possibility distributions are not the only mechanism to address incompleteness and parallel conceptual-mathematical approaches have been designed and tested.

Uncertainty has been faced through different concepts: capacity, belief, plausibility, possibility, and necessity among others. Besides their application potential is not diminished, they are *complete non-additive* and do not assume *self-duality*. These are bold differences to probability theory as imply an axiomatic change and incongruity with the *law of contradiction* and the *law of the excluded middle* as pointed by Guo, Guo, and Thiart (2010). Liu's credibility theory can face these shortcomings and find a direct relation to probability theory. Aforementioned affinity is desired due to the deep mathematical background of probability theory. Credibility theory has been improved by Love, Guo, and Li (2007) by adding hazard functions and even a clustering algorithm has been proposed by Rostam Niakan Kalhori, Fazel Zarandi, and Turksen (2014).

Beyond the mathematical background, the representation of uncertainty and human thinking can follow different premises as long as it reflects data nature or logical consistency. The understandability compliance proposed by Wijayasekara and Manic (2014) shows a very contrasting approach based on human interpretation of a fuzzy system. Similarly, Alikhademi and Zainudin (2014) tried to find fuzzy sets which are interpretable rather than precise as "the main role of fuzzy sets and MFs is transforming quantitative values to linguistic terms". In addition, Maisto and Esposito (2012) proposed a measure of distinguishability to enhance linguistic consistency by avoiding sets overlapping.

In the same vein, uncertainty measures optimality is application dependent. Consequently, many approaches are based on the principles stated by the ISO guide on metrology (Mauris, Lasserre, & Foulloy, 2001) instead of a traditional mathematical background: The measure must (a) characterize the dispersion (b) provide intervals of confidence, and (c) be easily propagated. Despite the available theoretical foundations, some methods have been derived regardless mathematical basis like the one proposed by Anoop, Rao, and Gopalakrishnan (2006), which finds a MF by piecewise-linear regression of a probability density distribution. This common framework relaxes the mathematical restrictions imposed to uncertainty representations and allows the possibility theory, for instance, to be the starting point for further developments; e.g., the proposal of veristic variables which can manipulate not one but many solutions to a given proposition (Yager, 2000a, 2000b). A veristic approach to classification has been reported by Younes, Abdallah, and Denœux (2010).

Until now, three main perspectives to represent uncertainty have been introduced: The first category can be described as *conceptual* as it models uncertainty regardless existent strong mathematical foundations and mostly based on reasonable logical assumptions. They can be mathematically associated with other theories but their principle is conceptual in general. The second one could be named *probability-based* as it tries to match the extensive basis of probability theory. Lastly, a third category can be composed by *interpretable* methods whose aim is to present the information in a human-understandable manner. Again, linkage to mathematical proposals can be made; however, their goal is not to fulfill data precise representation but its meaning.

Some conceptual methods can be found in literature: Wang and Mendel (1992) proposed to fill the universe of discourse with sigmoid and Gaussian MFs in an automatic way, satisfying the close-world assumption (Evaluated MFs at any point x must sum to one). Medaglia, Fang, Nuttle, and Wilson (2002) proposed to fit a histogram by the use of Bezier curves. Histogram usage was also considered by Masson and Denœux (2006) who found vertical simultaneous confidence intervals for each assumed certainty level, so the possibility distribution could be later computed through linear programming. Some other approaches seek the inclusion of conceptual benefits of aforesaid techniques, combined with other standard methods like Laanaya, Martin, Aboutajdine, and Khenchaf (2010) who used possibility and belief functions to enhance support vector classification. Different studies compare aforementioned techniques under a specific problematic like Baudrit and Dubois (2006) who try to bound undetermined probability distributions by using possibility and belief functions, as well as probability boxes.

Whenever a large amount of data is available, clustering methods provide an alternative for unknown distribution classification. Uncertainty is integrated to clustering assumptions so the results can deal with noise and membership sharing. Possibility was employed by Krishnapuram and Keller (1993) to enhance fuzzy c-means method, typicality is commonly addressed to evaluate membership based on distance measurement like by Setnes (2000), Pal, Pal, Keller, and Bezdek (2005) and Guarracino, Irpino, Jasinevicius, and Verde (2013). A similar approach can be found when the main aim of data evaluation is to derive fuzzy rules. This perspective pretends to find relations between domains as IF–THEN rules (Alikhademi & Zainudin, 2014; Kaya & Alhajj, 2004), or to describe their dependency in a regression fashion (Dickerson & Kosko, 1996; Kosko, 1994).

Possibility–probability transformations have also been of great interest in current research as some principles can provide linkages between both approaches. Basic restrictions are known to be the consistency principle $\pi_A(x) \geqslant p_A(x)$, the order preservation principle (functions shapes must be similar), and the maximum specificity principle (a less spread sample leads to more specific information) (Hou & Yang, 2010; Mauris, 2008). More complex approaches have found a relation of possibility theory to upper and lower probability distributions for some probability family as presented by Mauris (2011). Consistency of probability confidence with possibility distribution description of α-cuts has been elaborated by Dubois, Foulloy, Mauris, and Prade (2004). Some comparative studies have been also presented like the one by He and Qu (2007).

In this paper, a new reasoning on the generation of membership functions is presented. It is derived from a conceptual approach to the *degree of accomplishment* associated to how easily an element within a fuzzy set can accomplish its label significance on its own regarding other options (other elements). The main difference to existing methods based on similar principles is that this degree is not only useful for intra-set characteristics, but also for inter-set relations. As a result, a subnormal function is derived whose normal-inverse reveals how difficult it is for that set to represent a sure assumption. The relation of this approach to certainty is also explained, and some linkages to possibility theory are also provided.

Our proposal is expanded by considering Gaussian MFs in univariate and bivariate domains. Calculation of certainty is derived and numerically exampled. This calculations are also related to the fuzziness of each set, so a connection to *specificity* is made, together to a discussion about data importance and noise rejection. The presentation of this approach in the sequel is mainly related to feasibility, certainty, possibility, and fuzziness through data dispersion knowledge.

2. Theory

2.1. Uncertainty basis

In order to make a clear distinction of the proposed approach to uncertainty representation, some of the foundations of possibility