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Fuzzy Sets and Systems 153 (2005) 181–194
FUZZY sets and systems
www.elsevier.com/locate/fss

On the symmetric difference of fuzzy sets

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Received 11 June 2004; received in revised form 9 February 2005; accepted 9 February 2005
Available online 17 March 2005

Abstract

By means of techniques arising either in the field of functional equations or in lattice theory we study some models for the symmetric difference of fuzzy sets when mixing connectives and in a single theory. In doing this some new functional expressions are obtained and some standard models used in the literature are reviewed.

Keywords: Symmetric difference; t-norms; t-conorms; Strong negations; Pexider equations; Metrics

1. Introduction

Given a universe X, in the Boolean algebra $(P(X), \cup, \cap, ^c)$ of its subsets one may consider several equivalent expressions to represent the *symmetric difference* $A\triangle B$ of two subsets of X, e.g.

$$A\triangle B = (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \cap (A \cap B)^c.$$

Although George Boole [5] did not make it explicit, he always considered 'x or y' $(x + y)$ when classes x and y have nothing in common. Hence, Boole always considered $x + y = x\Delta y$, since in Boolean algebras this follows from $x + y = x\Delta y + x \cdot y$ when $x \cdot y = 0$. From the very beginning of the calculus of classes, the symmetric difference, the representation of the exclusive "or", was an important operation that, until now, was not studied in detail within fuzzy sets.

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1 This paper is partially funded by the European Project OmniPaper (IST-2001-32174).

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doi:10.1016/j.fss.2005.02.009

When in Fuzzy Logic one considers conjunctions, disjunctions and negations described, respectively, by means of t-norms, t-conorms and strong negations in the unit interval [0, 1] (see e.g. [2,8,10]) then one can model a symmetric difference by means of expressions like $S(T(x, N(y)), T(y, N(x)))$ or $T(S(x, y), N(T(x, y)))$ generalizing the above set's definitions. This has been the case in the literature of the field because the interest of symmetric differences has been motivated by the introduction of metrics in fuzzy sets or their applications, e.g. to similarity or matching relations, image retrieval, etc. (see, e.g. [4,7,11–13], etc.).

Our aim in this paper is to start with a very general definition of a symmetric difference operator and then to proceed to characterize several functional models and to compare the corresponding results with the expressions being used up to now. In doing this some surprising conclusions arise. Some of our models will be of Pexider type [1], i.e. involving as many different t-norms, t-conorms and negations as possible, so we cover a very general framework going further than the usual models where a given triplet (T, S, N) is fixed in advance.

Nevertheless, it is also important to know in which standard theories of fuzzy sets $([0, 1]^X, T, S, N)$, a symmetric difference $\hat{\triangle} : [0, 1]^X \times [0, 1]^X \to [0, 1]^X$ can be defined in such a way that, being consistent with the classical case, verifies, as much as possible, properties of the sets' symmetric difference. Consistency means that if μ, σ are in $\{0, 1\}^X$ also $\mu\hat{\Delta}\sigma$ is in $\{0, 1\}^X$, and coincides with the symmetric difference of the corresponding sets, i.e. with the membership function of $\mu^{-1}(1)\Delta\sigma^{-1}(1)$.

In what follows, only the functionally expressible case, $(\mu\hat{\Delta}\sigma)(x) = \Delta(\mu(x), \sigma(x))$, for all μ, σ in $[0, 1]^X$, all x in X, and $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ will be considered. To be consistent with the classical case, operations Δ in [0,1] do verify $\Delta(0, 0) = \Delta(1, 1) = 0$ and $\Delta(0, 1) = \Delta(1, 0) = 1$.

From now on I will denote the closed unit interval [0, 1].

Sections 2–5 consider the general case where several connectives are taken into account by generalizing the symmetric difference in Boolean algebras (sets) (see e.g. [3]). Sections 6 and 7, consider the case of a single theory $([0, 1]^X, T, S, N)$ but looking at it from the point of view of those lattices where a symmetric difference may be defined.

2. On symmetric difference operators by mixing connectives

We start with the following.

Definition 1. A two-place function $\triangle$ from $I \times I$ into I will be called a *symmetric difference operator* if the following conditions are satisfied for all a in I:

(i) $\triangle(a, 0) = \triangle(0, a) = a$;
(ii) $\triangle(a, a) = 0$;
(iii) $\triangle(a, 1) = \triangle(1, a) = N(a)$, where N is a strong negation.

We consider that the three properties above are the basic requirements that we can expect from a symmetric difference operator, i.e. we only fix the behaviour of $\triangle$ on the boundaries of $I \times I$ and its diagonal. Obviously this behaviour is consistent with the classical case.

It is interesting to remark that while the symmetric difference of crisp sets is an associative binary operation, all operators described in Definition 1 *cannot be associative*. To see this note that since we