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On the relation between equations with max-product composition and the covering problem

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Abstract

Systems of equations with max-product composition are considered. It is shown that solving these equations is closely related with the covering problem, which belongs to the category of NP-hard problems. It is proved that minimal solutions of equations correspond to irredundant coverings. In terms of the covering problem the conditions of compatibility of equations, of redundancy of equations, of uniqueness of solution, of uniqueness of minimal solution are determined. Concepts of essential, non-essential, semi-essential and super-essential variables are suggested. Ways of simplification of a covering problem and methods of its solving are considered.

Keywords: Fuzzy relations; Fuzzy relation equations; Inverse problem; Max-product composition; Minimal solutions; Covering problem; NP-hard problems; NP-complete problems

1. Introduction

The systems of equations with max-product composition arise in the inverse problems for finite fuzzy sets and relations [1,4–6]. These problems are formulated as matrix equations of the kind $X \circ A = B$ or $A \circ X = B$, where X, A and B are matrices whose elements are defined on [0,1], and the symbol '$\circ$' denotes max-product composition, meaning that the element b_{ij} of the composition of matrices X and A is defined as $b_{ij} = \max\{x_{ik}a_{kj}\}$ [1,6]. Obviously, any matrix equation of this type, being decomposed

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into elements b_{ij}, may be presented as a system S of equations of the following kind:

$$\begin{aligned}
&x_1a_{11} \vee \cdots \vee x_ia_{i1} \vee \cdots \vee x_ma_{m1} = b_1,\\
&\cdots\\
&x_1a_{1j} \vee \cdots \vee x_ia_{ij} \vee \cdots \vee x_ma_{mj} = b_j,\\
&\cdots\\
&x_1a_{1n} \vee \cdots \vee x_ia_{in} \vee \cdots \vee x_ma_{mn} = b_n,
\end{aligned}$$

where $x_i, a_{ij}, b_j \in [0, 1]$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$, and the symbol of disjunction '$\vee$' denotes the operation of taking maximum. Let s_j denote the jth equation of the system S. Let us say that variable x_i *enters* into equation s_j (equation s_j *contains* x_i) if $a_{ij} \neq 0$. A *solution* of a system S is a combination of values of variables x_i, satisfying all its equations. A system S is *compatible* if it has at least one solution.

We consider the complete set of solutions of a system S and, in particular, the set of all minimal solutions. Earlier [1,6] it was noted that this problem has a combinatorial character. But what sort of a combinatorial problem it is, a new or a famous one, and what is its computational complexity have never been identified. We show that this problem is a well-known covering problem, properly reformulated.

2. Reduction of the equations to the covering problem

Obviously, if $b_j = 0$ $(1 \leqslant j \leqslant n)$, then the values of all variables, which appear in equation s_j, must be equal to 0. Hence, instead of the system S, we can consider a simplified system S', in which all such equations, and all variables they contain are eliminated. Any solution of a system S can be obtained from a solution of the system S' by the addition of zero values of eliminated variables. For simplicity, let us assume further that such a transformation has been done and $b_j > 0$ for any j.

If $R = (r_1, \ldots, r_i, \ldots, r_m)$ is a solution of a system S, then for each equation s_j there exists such a value r_i of some variable x_i, that $r_ia_{ij} = b_j$. Let us say that r_i is a *realizing* value for s_j in R and that s_j is *realized* by r_i in R.

For a realizing value r_i, the equality $r_i = b_j/a_{ij}$ holds ($a_{ij} \neq 0$ if x_i enters s_j) and since $r_i \leqslant 1$, the inequality $b_j/a_{ij} \leqslant 1$ holds as well. Since, by assumption, all $b_j > 0$, all *realizing values are positive*.

Let us call a variable x_i *essential* if $b_j/a_{ij} \leqslant 1$ for some j, and *non-essential* otherwise. A non-essential variable x_i cannot have realizing value r_i in the solutions, since for any equation s_j, containing x_i, from $r_i \leqslant 1$ and $b_j/a_{ij} > 1$ follows $r_ia_{ij} < b_j$. In other words, non-essential variables have no influence on satisfiability of equations.

So the equations of a system S can be satisfied only by essential variables. The presence of essential variables is a necessary condition of compatibility of a system S. If a system S has no essential variables, then it is non-compatible. But if a system S has essential variables, then it can be both compatible and non-compatible. The necessary and sufficient conditions of compatibility of a system S are formulated in Theorem 2.

Let x_i be an essential variable and $\hat{x}_i = \min(b_j/a_{ij})$ for all equations s_j containing x_i. Let us name $\hat{x}_i$ as the *base value* of x_i. Let us say that $\hat{x}_i$ *belongs* to an equation s_j (or s_j *possesses* $\hat{x}_i$) if $\hat{x}_i = b_j/a_{ij}$ (i.e. $\min(b_j/a_{ij})$ is achieved on s_j). A base value $\hat{x}_i$ can belong to several equations. An equation can possess base values of several variables. As all $b_j > 0$, *all base values are positive*.