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Fuzzy measures and integrals

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Abstract

We present fuzzy measures and fuzzy integrals as special poset homeomorphisms. Besides general fuzzy integrals, regular fuzzy integrals are introduced and some of their properties are discussed. State of art of fuzzy measure and fuzzy integral theory are briefly summarized and some further streaming is sketched.

Keywords: Fuzzy measure; Fuzzy integral; Regular fuzzy integral

1. Introduction

Classical measure theory, see, e.g., [12], is based on a measure space $(X, \mathcal{A}, m)$, where $\mathcal{A}$ is a σ-algebra of subsets of a non-empty set X and m is a non-negative σ-additive real set function defined on $\mathcal{A}$. Recall that $(X, \mathcal{A})$ is called a measurable space, sets from $\mathcal{A}$ are called events and m is called a measure. In special case when $m(X) = 1$ is required, m is called a probability measure. The Lebesgue integral L related to a measure space $(X, \mathcal{A}, m)$ is a special functional acting on measurable real functions on X which can be viewed as a monotone extension of the measure m (i.e., for all $A \in \mathcal{A}$, $L(\mathbf{1}_A) = m(A)$). Classical measure and integral theory was extended, generalized and deeply examined in many directions. For an exhaustive state-of-art overview we recommend the recent handbook [22] edited by Pap.

In this contribution we will deal with special generalization of measure theory unifying the notions of measures and integrals. Our main aim is to clarify the notion of fuzzy integral.

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2. Fuzzy measures

The (σ-) additivity of a measure m is rather restrictive property and it does not allow to model several properties of the real world, for example, the interaction. This fact was observed by many researchers and has led to several modifications and/or generalizations of the additivity property. Recall, e.g., Vitali [27] in 1925 or Choquet [4] in 1953. One of the most influential works in this direction was the thesis [25] of Sugeno, who introduced fuzzy measures and the related integral, which is even named after him. Fuzzy measures and related integrals (the Choquet integral, the Sugeno integral, ...) were discussed in many papers and in several monographs. We only recall here the monographs of Wang and Klir [29] from 1992, Denneberg [5] from 1994, Pap [21] from 1995, and the edited volume of Grabisch et al. [9] from 2000. In these works one can find all definitions and results which are explicitly not included in this paper.

Observe that neither for fuzzy measures nor for fuzzy integrals there is a definition covering all works in the field. To reduce the diversity, we will restrict our considerations to the range [0, 1] only. Observe that $([0, 1], \leqslant)$ is a poset (even a chain) with top element 1 and bottom element 0. Fuzzy measures discussed in [25,29,21,9] act on events (measurable sets), i.e., on the poset $(\mathcal{A}, \subseteq)$ with top element X and bottom element $\emptyset$. Therefore, fuzzy measures can be treated as special homeomorphisms of bounded posets.

Definition 1. Let $(U, \preceq)$ and $(V, \leqslant)$ be two bounded posets with top elements $\mathbf{1}_U$ and $\mathbf{1}_V$ and bottom elements $\mathbf{0}_U$ and $\mathbf{0}_V$, respectively. A mapping $\varphi : U \to V$ is called the poset homeomorphism if it preserves the ordering and the bounds, i.e., if

(i) $a, b \in U$, $a \preceq b$ implies $\varphi(a) \leqslant \varphi(b)$,
(ii) $\varphi(\mathbf{1}_U) = \mathbf{1}_V$ and $\varphi(\mathbf{0}_U) = \mathbf{0}_V$.

In the light of Definition 1, fuzzy measures on $(X, \mathcal{A})$ are exactly poset homeomorphisms between $(\mathcal{A}, \subseteq)$ and $([0, 1], \leqslant)$. Several special types of fuzzy measures, such as belief and plausibility measures, possibility and necessity measures, submeasures, supermeasures, etc., are characterized by the corresponding properties. Moreover, dual fuzzy measures can be introduced as dual poset homeomorphisms providing the existence of order-reversing involutions for each involved poset, i.e., for a given fuzzy measure $m : \mathcal{A} \to [0, 1]$, its dual $m^d : \mathcal{A} \to [0, 1]$ is given by $m^d(A) = 1 - m(A^c)$ (observe that the set complement is an order-reversing involution on $(\mathcal{A}, \subseteq)$ while $n : [0, 1] \to [0, 1]$, $n(x) = 1 - x$, is an order-reversing involution on $([0, 1], \leqslant)$).

3. Fuzzy integrals

For a given measurable space $(X, \mathcal{A})$, let $\mathcal{F}$ be the set of all measurable $X \to [0, 1]$ mappings. Evidently, $\mathcal{A}$ can be embedded into $\mathcal{F}$ by means of characteristic functions. Moreover, the standard partial order $\leqslant$ of functions makes $(\mathcal{F}, \leqslant)$ to be a bounded poset with top element $\mathbf{1} = \mathbf{1}_X$ and bottom element $\mathbf{0} = \mathbf{1}_\emptyset$. Any poset homeomorphism $M : \mathcal{F} \to [0, 1]$ can be understood as a monotone extension of a fuzzy measure $m : \mathcal{A} \to [0, 1]$ given by $m(A) = M(\mathbf{1}_A)$, $A \in \mathcal{A}$. Observe that any “fuzzy integral” known from the literature is a special poset homeomorphism of the above type (providing the restriction of integrands to measurable [0, 1]-valued functions).