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Remarks on basics of fuzzy sets and fuzzy multisets

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Abstract

Two remarks on basic aspects of fuzzy sets are given. They are notation for the α-cuts and a commutative property in the extension principle. Application of the commutative property to fuzzy multisets is moreover considered.

Keywords: Commutative properties; Extension principle; Fuzzy multiset

1. Introduction

Two basic points on fuzzy sets are remarked in this small report; they are notation for α-cuts and commutative properties between set operations, and the extension principle. We emphasize the usefulness of a commutative property between a set operation and an α-cut.

Application of the commutativeness to fuzzy multiset image is also described where basic operations and a multiset image are defined.

2. The α-cuts

Good notations will lead to deeper understanding of a theory. In the author's opinion there are rooms of improving notations in fuzzy sets. An example is the α-cuts.

The strong α-cut is the crisp set where the membership is 'greater than' the value of α whereas the weak α-cut is the crisp set where the membership is 'greater than or equal to' the value of α.

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Assume that X is a metric space and membership function $\mu_A(x)$ is continuous as many real examples suppose. It is now easy to see that the strong α-cut gives an open set whereas the weak α-cut provides a closed set. From this observation good notations for α-cuts are $]A[_\alpha$ for a strong α-cut, and $[A]_\alpha$ for an weak α-cut.

It is well-known that α-cuts commute with the basic set operations $\cup$ and $\cap$ such as $]A \cup B[_\alpha =]A[_\alpha \cup]B[_\alpha$, $[A \cap B]_\alpha = [A]_\alpha \cap [B]_\alpha$, etc. Such commutative properties are very important in understanding an old theory and in developing a new theory.

3. Extension principle

Let Y be another set and $f: X \to Y$. Most literature describes the extension principle as $\mu_{f(A)}(y) = \sup_{x \in f^{-1}(y)} \mu_A(x)$, etc. The author does not oppose this definition, but there is more to be considered in relation to the crisp definition of the set image.

Let us consider a simple crisp example. We use a simple function $\{\cdot\}$: $x \stackrel{\{\cdot\}}{\mapsto} \{x\}$ that maps an element to a set that consists of that element alone. Then we have $f(\{x\}) = \{f(x)\}$ from the standard crisp definition. Thus f and $\{\cdot\}$ are commutative. Now, suppose $B = \{x, x'\} = \{x\} \cup \{x'\}$. We have $f(B) = f(\{x\} \cup \{x'\}) = \{f(x)\} \cup \{f(x')\}$. Namely the map $\stackrel{\{\cdot\}}{\mapsto}$ and the operation $\cup$ commute with $f(\cdot)$.

More generally, we note $f(A) = \bigcup_{x \in A} \{f(x)\}$. This equation is applicable to fuzzy set images.

We return to the first map $x \mapsto \{x\}$ and generalizes it to fuzzy sets. For this purpose consider a fuzzy point $(x, \mu_A(x))$ and $(x, \mu_A(x)) \mapsto \{(x, \mu_A(x))\}$. The last $\{(x, \mu_A(x))\}$ is a fuzzy set which has nonzero membership at x alone. We can write $A = \{ (x, \mu_A(x)) \in X \times [0, 1] \} = \bigcup_{x \in X} \{(x, \mu_A(x))\}$. We now have $f(A) = \{ (f(x), \mu_A(x)) \in Y \times [0, 1] \} = \bigcup_{x \in X} \{(f(x), \mu_A(x))\}$. The last expression has redundancy when $f(x) = f(x')$ for $x \neq x'$. Naturally the union operation is taken for the redundant case and we have the image of a fuzzy set. In other words, the rewriting rule $\cdot \mapsto f(\cdot)$ is used in common: $A = \bigcup_{x \in A} \{x\} \Rightarrow f(A) = \bigcup_{x \in A} \{f(x)\}$ for a crisp set, and $A = \bigcup_{x \in X} \{(x, \mu_A(x))\} \Rightarrow f(A) = \bigcup_{x \in X} \{(f(x), \mu_A(x))\}$ for a fuzzy set. To note the commutative property thus appears useful.

We now have a new problem, however. The above definition using fuzzy points is not equivalent to the extension principle. To see this, notice that $[f(A)]_\alpha \neq f([A]_\alpha)$ in general: the weak α-cut and the functional image is not commutative. In contrast, it is easily seen that $[f(A)]_\alpha = f([A]_\alpha)$ from the above definition, since $y \in [f(A)]_\alpha$ means there exists an x such that $y = f(x)$ and $x \in [A]_\alpha$, which does not hold from the extension principle.

Why does such a difference occur? The reason is that infinite union of fuzzy points may lead to a 'half-open' fuzzy point. For example, $\bigcup_{1 \leqslant n < \infty} (x, [0, 1 - 1/n]) = (x, [0, 1[)$. Thus the infinite union fails to have a membership. On the other hand, if we use an identity map $I(x) = x$, the extension principle leads to $I(\bigcup_{1 \leqslant n < \infty} (x, [0, 1 - 1/n])) = (x, [0, 1])$.

Such a difficulty has been known by theoretical researchers, and we may have different attitudes.

(i) Satisfy with finite operations. Then the commutative property and the extension principle are consistent.
(ii) Give up the commutative property; it is enough to have the extension principle.
(iii) Analyse the cases when the commutative property holds or not, and derive conditions for the commutativeness.