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Fuzzy Sets and Systems 155 (2005) 18–25

FUZZY sets and systems

www.elsevier.com/locate/fss

Non-disturbance for fuzzy quantum measurements

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Available online 2 June 2005

Abstract

We consider three criteria for describing non-disturbance between quantum measurements. While previous discussions of these criteria considered sharp measurements, we shall treat more general measurements that may be unsharp (or fuzzy). It has been shown that in the sharp case, the three criteria are equivalent to compatibility of the measurements. We shall show that only the third criterion is equivalent to compatibility for the general fuzzy case. Moreover, the first two criteria are not symmetric relative to the two measurements.

Keywords: Quantum measurements; Hilbert space; State; Conditional probability

1. Introduction

Kirkpatrick [9] has recently discussed three ways of describing non-disturbance between quantum measurements. The first two are due to Lüders [10] and the third to Davies [5] and Kirkpatrick [8]. Letting X and Y be two quantum measurements, the three non-disturbance criteria are given roughly as follows.

(1) The probability of an established value of Y is unchanged by the later occurrence of a value of X.
(2) The probability of occurrence of a Y value is unchanged by a preceding execution of X.
(3) If p and q are X and Y values, respectively, then the probability of p followed by q coincides with the probability of q followed by p.

We said that the criteria are given roughly because the English language can be imprecise and ambiguous. We shall later translate these criteria into precise mathematical language and prove results about

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doi:10.1016/j.fss.2005.05.009

them. It is appropriate that these criteria are phased in terms of probabilities because statistical results are what are observed in the laboratory. Kirkpatrick showed that (1), (2) and (3) are equivalent to the compatibility (commutativity) of X and Y and hence are equivalent to each other. However, Kirkpatrick only considered sharp quantum measurements which are described by projection-valued measures. In the present paper we shall generalize his results to measurements that may be unsharp (or fuzzy) and these are described by positive operator-valued (POV) measures. Unsharp measurements are quite important in quantum measurement theory [3,2,5] and in applications such as quantum computation and quantum information [11].

Unlike the sharp case, we shall show that (1), (2) and (3) are not equivalent for general measurements. Only criterion (3) is equivalent to the compatibility of X and Y. Although criterion (1) implies compatibility it is not appropriate for the general case because it also implies that Y is sharp. Compatibility implies (2) but the converse does not hold. Although they are not explicitly symmetric relative to X and Y, the three criteria are implicitly symmetric in the sharp case because they are equivalent to the symmetric relation of compatibility. In the general fuzzy case, only (3) is symmetric in X and Y. This further justifies Kirkpatrick's choice of (3) for his discussion of noncompatibility in classical probability theory [9]. As in Kirkpatrick [9] and Lüders [10], we shall only consider discrete quantum measurements.

2. Measurements and probabilities

Let H be a complex Hilbert space representing the state space of a physical system S. Let $\mathcal{P}(H)$ be the set of projections on H and $\mathcal{E}(H)$ the set of operators on H satisfying $0 \leqslant A \leqslant I$. The elements of $\mathcal{E}(H)$ are called *effects* and represent yes–no measurements on S that may be unsharp, while the elements of $\mathcal{P}(H)$ represent sharp effects [4,6,7]. Let $\mathcal{D}(H)$ be the set of positive trace class operators on H with unit trace. The elements of $\mathcal{D}(H)$ are called *density operators* and represent the set of *states* of S. When S is in state $W \in \mathcal{D}(H)$, the probability that the effect $A \in \mathcal{E}(H)$ is observed (or has value yes) is given by $p_W(A) = \mathrm{tr}(AW)$, and if A is observed then the post-measurement state becomes

$$A^{1/2}WA^{1/2}/\mathrm{tr}(AW), \tag{2.1}$$

whenever $\mathrm{tr}(AW) \neq 0$. Of course, if $\mathrm{tr}(AW) = 0$ then with certainty A will not be observed in state W so the fact that (2.1) is meaningless in this case is of no consequence.

For $A, B \in \mathcal{E}(H)$, the *conditional probability* that B is observed given that A has been observed is defined by

$$p_W(B \mid A) = \mathrm{tr}(BA^{1/2}WA^{1/2})/\mathrm{tr}(AW), \tag{2.2}$$

whenever $\mathrm{tr}(AW) \neq 0$ [4,7]. Notice that (2.2) follows in a natural way from (2.1). For $A, B \in \mathcal{E}(H)$ we use the notation $A\&B$ to denote the effect in which A is performed first and then B is performed next. We call $A\&B$ "A and then B." Then in a natural way we have that

$$p_W(A\&B) = p_W(A)p_W(B \mid A) = \mathrm{tr}(BA^{1/2}WA^{1/2})$$

and

$$p_W(C \mid A\&B) = \mathrm{tr}(CB^{1/2}A^{1/2}WA^{1/2}B^{1/2})/\mathrm{tr}(BA^{1/2}WA^{1/2}), \tag{2.3}$$

whenever $\mathrm{tr}(BA^{1/2}WA^{1/2}) \neq 0$.