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The pseudo-linear superposition principle for nonlinear partial differential equations and representation of their solution by the pseudo-integral

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Abstract

In this paper, for a pseudo-linear partial differential equation of the nth order the pseudo-superposition principle is proved, which means that a pseudo-linear combination of solutions of the equation is again a solution of this equation. Especially, this principle is proved for some equations of the second order imposing weaker conditions. In addition, theorems giving the representation of the solution of Cauchy problem by pseudo-integral are obtained.

Keywords: Generalized pseudo-addition; Generalized pseudo-multiplication; Δ-derivative; Δ-integral; Pseudo-linear superposition principle

1. Introduction

Nonlinearity often occurs in the complex models. It is reflected in nonlinear equations which describe these models [1,2,4,6,12]. We consider some operations which are different from the usual addition and multiplication (we shall call such operations pseudo-addition $\oplus$ and pseudo-multiplication $\odot$), see [7,11]. These operations have some canonical representations which are very useful in different applications. Based on such representations, some mathematical tools were developed for different purposes, which we shall call pseudo-analysis [7,11,14,17,3,13]. Pseudo-analysis has applications in different fields, e.g., measure theory, integration, integral operators, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc.

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The pseudo-operations $\oplus$ and $\odot$ are commutative, associative, nondecreasing ($\odot$ is positively non-decreasing) and they have a neutral element (**0** and **1**, respectively) over a corresponding interval $I \subseteq [-\infty, \infty]$. If the operation $\odot$ is distributive with respect to the operation $\oplus$ and **0** is an annihilator for the operation $\odot$, we call the triplet $(I, \oplus, \odot)$ a real semiring, for example $([-\infty, \infty], \max, \min)$ or $([-\infty, \infty), \max, +)$. In this paper we consider the generalized pseudo-operations introduced in [15], which are not necessarily commutative and associative. In Section 2, for a special class of generalized pseudo-operation three—parameter pseudo-operations—we introduce the notions of pseudo-integral and pseudo-derivative and their properties with respect to these operations.

In Section 3, we consider a general pseudo-partial differential equation with respect to pseudo-derivative and examine in the basic Theorem 3.1 under which conditions on operations the pseudo-linear principle holds for the second order partial differential equation. The nonlinear partial differential equations of the wave equation type, the heat equation type and Laplace equation type are discussed here.

In Section 4, we consider the Cauchy problem for the second order partial differential equations and we give a representation of their solutions based on the pseudo-integral.

2. Preliminary notions and notations

We recall the notion of the generalized pseudo-operation defined in [15].

Definition 2.1. We call real operations $\oplus$ and $\odot$ generalized pseudo-addition and generalized pseudo-multiplication from the right (or from the left), if they satisfy the following conditions:

(i) $\oplus$ and $\odot$ are functions from $C^2(\mathbb{R}^2)$;
(ii) the equation $t \oplus t = z$ for given z is uniquely solvable;
(iii) $\odot$ is right (left) distributive over $\oplus$:

$$(D_r)(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z) \quad (D_l)\ z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y).$$

Let f_1 and f_2 be two functions defined on the interval $[c, d] \subset \mathbb{R} \cup \{\pm\infty\}$ and with values in $[a, b]$. Then, we define for any $x \in [c, d]$

$$(f_1 \oplus f_2)(x) = f_1(x) \oplus f_2(x),$$

$$(f_1 \odot f_2)(x) = f_1(x) \odot f_2(x)$$

and for any $\lambda \in [a, b]$

$$(\lambda \odot f_1)(x) = \lambda \odot f_1(x).$$

We consider the following special class of operations introduced in [16], see also [15], by

$$u \oplus v = k^{-1}(\varepsilon_1 k(u) + \varepsilon_2 k(v)) \tag{1}$$

and

$$u \odot v = k^{-1}(k^{\delta}(u) \cdot k(v)), \tag{2}$$