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Conditional probability on σ-MV-algebras☆

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Abstract

Using the Loomis–Sikorski representation of σ-MV-algebras and Dedekind σ-complete ℓ-groups, a conditional probability and expectation on these structures are defined as a natural generalization of the usual Kolmogorovian conditioning.

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1. Introduction

The aim of this paper is to apply generalizations of the classical Loomis–Sikorski theorem to σ-MV-algebras and Dedekind σ-complete ℓ-groups [8,9,16] in the development of probability theory on MV-algebras and related structures, see [1,19,20] for an introduction to the problematic. In particular, we concentrate our attention on the important concepts of conditional probabilities and conditional expectations. First step in this direction was already made in [14], where σ-MV-algebras with product were considered. In the present paper, we consider general σ-MV-algebras and Dedekind σ-complete ℓ-groups, and find a natural generalization of the Kolmogorovian theory of conditional probability and conditional expectations to these structures.

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2. Preliminary results

In this sequel, we summarize some basic facts on MV-algebras and probability theory on MV-algebras. See [6,9] for further results on MV-algebras and [19] for an exposition of probability on MV-algebras. We recall that MV-algebras were originally introduced by Chang as an algebraic basis of many-valued logic [4,5].

An *MV-algebra* is an algebra $(M; \oplus, ', 0)$, where $\oplus$ is an associative and commutative binary operation on M having 0 as a neutral element, a unary operation $'$ is involutive and such that $a \oplus 0' = 0'$ for all $a \in M$, and, in addition, the Łukasziewicz identity $a \oplus (a \oplus b')' = b \oplus (b \oplus a')'$ is satisfied for all $a, b \in M$. If no ambiguity may arise, we say that M is an MV-algebra.

An additional constant 1 and two binary operations $\odot$ and $\ominus$ are defined as follows: $1 = 0'$, $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$. A partial order is defined on M by $a \leqslant b$ iff $a' \oplus b = 1$. In this ordering, M is a distributive lattice with 0 as the smallest and 1 as the greatest elements, respectively. The lattice operations are defined as follows: $a \vee b = a \oplus (a \oplus b')'$, $a \wedge b = (a' \vee b')'$. MV-algebras satisfying the additional axiom $a \oplus a = a$ for all a coincide with Boolean algebras. A *sub-MV-algebra* of M is a subset N of M containing the neutral element 0 of M, closed under the operations of M and endowed with the restriction of these operations to N. An MV-algebra is *σ-complete*, or a *σ-MV-algebra* if every nonempty countable subset of M has a supremum in M.

For any MV-algebra M, the set $\mathcal{B}(M) := \{a \in M : a \oplus a = a\}$ of all idempotents of M, with the operations inherited from M, is a Boolean algebra, the largest Boolean subalgebra of M. If M is a σ-MV-algebra, then $\mathcal{B}(M)$ is a Boolean σ-algebra.

Let M and N be MV-algebras. A mapping $h : M \to N$ is a *homomorphism* if $h(0) = 0$, $h(a \oplus b) = h(a) \oplus h(b)$, and $h(a') = h(a)'$, for any $a, b \in M$. If M and N are σ-MV-algebras, a homomorphism $h : M \to N$ is a *σ-homomorphism* if

$$h\left(\bigvee_{i \in \mathbb{N}} a_i\right) = \bigvee_{i \in \mathbb{N}} h(a_i).$$

An *ideal* of an MV-algebra M is a subset I of M such that $0 \in I$, $a \oplus b \in I$ whenever $a, b \in I$ and $a \leqslant b$ with $b \in I$ implies $a \in I$. An ideal I of M is called *maximal* if the only ideal strictly containing I is the improper ideal M. Let $\mathcal{I}(M)$ and $\mathcal{M}(M)$ denote the set of all ideals and the set of all maximal ideals of M, respectively. The space of all maximal ideals $\mathcal{M}(M)$ is nonempty, and endowed with the so-called *spectral topology* formed by the sets of the form $O_J = \{K \in \mathcal{M}(M) : K \not\supseteq J\}$ for all $J \in \mathcal{I}(M)$, it becomes a compact Hausdorff space.

We say that $(M; \oplus, \cdot, ', 0)$ is an *MV-algebra with product* if $(M; \oplus, ', 0)$ is an MV-algebra and a product $\cdot$ is a commutative and associative binary operation on M such that for all $a, b, c \in M$:

(1) $1 \cdot a = a$;
(2) $a \cdot (b \oplus c) = (a \cdot b) \oplus (a \cdot c)$.

It is straightforward to see that 0 acts as the zero element of M with respect to the product, and that the product is monotone, that is, if $a \leqslant b$ then $a \cdot c \leqslant b \cdot c$ for all $c \in M$.

For any lattice ordered group (ℓ-group, for short) G, an element $u \in G$ is said to be a *strong unit* of G if for every $g \in G$ there is an integer $n \geqslant 1$ such that $nu \geqslant g$ [10]. By a *morphism* $\phi : (G, u) \to (G', u')$ we mean a group homomorphism $\phi : G \to G'$ that also preserves the lattice structure and satisfies the