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The minimal mathematical structure for a synchronic approach to fuzzy set theory

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Abstract

The paper gives a concrete presentation of the minimal structure on the monoid of an external subset classifier of a universe of usual sets with Zadeh–Goguen L-fuzzy subsets. The result was first investigated by the first author, with the aim of providing a new insight into the synchronic approach to fuzzy set theory. The extensive use of conceptual mathematics (i.e. category theory) in the original paper, restricts the audience of readers of this analysis to persons who are already acquainted with conceptual mathematics. The aim of this paper is to avoid as much as possible the use of functorial terminology in the presentation of the minimal mathematical structure on the set of truth values, for a synchronic approach to fuzzy sets. By doing so, we hope to widen the audience of scientists interested in the foundation of Zadeh–Goguen L-fuzzy sets and to provide a new way of looking at the synchronic approach to fuzziness.

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1. Introduction

Initiated by Zadeh [18] and slightly generalized by Goguen [2] as a new basis of mathematical reasoning which could replace Cantor's rigid set theory [1], fuzzy set theory was quickly beset by critical opinions relative to its status as a new foundation for mathematics. Despite such negative critical opinions, however, fuzzy set theory attracted and continues to attract scientists from many fields of human knowledge (see Section 1 of Goguen [4]). It was indeed accepted early, in any case that L-fuzzy sets extend the notion of membership relation of a subset of a carrier set, through being a characteristic function defined on the carrier set. This is explicitly reflected later in [5] where the word "fuzzy set" is replaced by "fuzzy subset". This change emphasizes the synchronic approach to fuzzy sets theory.

In trying to establish a link between fuzzy set theory and Lawvere–Tierney's conceptual axiomatic theory of sets [7,16] (models of the theory are called toposes or topoi), Negoita [13] emphasizes two approaches to fuzzy set theory. The diachronic approach considers an L-fuzzy set as a dynamic process along the set L of various levels of knowledge of members of the set. From this point of view, L-fuzzy sets are basic objects of interest and usual sets are particular cases of these generalized sets. Goguen's papers [3,4] can be classified as belonging to this approach. The synchronic approach to fuzziness views fuzzy sets as a static process where an L-fuzzy set is considered as a generalized characteristic function of a subset of an usual set, the carrier of the subset. In this approach, usual sets are ipso facto accepted as basic objects and the membership relation is fuzzified.

A conceptual analysis of the synchronic approach to fuzziness was initiated by the first author [10]. The main question "How fuzzy sets can be viewed as subsets of usual sets?" was raised at Tizi–Ouzou (compare Ref. [8]). The basic ingredient of the analysis is the assumption that an L-fuzzy set is a subset of a set in a universe of usual sets where mappings between sets are generalized. Conceptual motivations for this point of view can be found in this paper; the gist is

> *A set X in a virtual universe* (*to be found*!) *for which subsets of a set X are L-fuzzy sets of carrier X is viewed in the universe of usual sets and mappings as $X \times L^*$ where L^* is the monoid of nonzero truth values of the new universe of L-fuzzy subsets*.

In Section 2, we explain briefly how the concept of a subset of an usual set depends on what is adopted as a mapping between sets. Models of the synchronic approach to fuzzy set theory are perceived as universes of usual sets for which a subset of a set X is an L-fuzzy set of carrier X. The concept of an L^*-mapping between sets, for a monoid L^*, is introduced here as a natural extension of the usual mapping between sets.

Given a monoid L^* for which we use the multiplicative notation (1 denotes the unit of L^*), the next section presents what shall be a natural extension of usual subsets by L-fuzzy subsets and establishes the following conditions on the monoid L^* as necessary conditions to fulfill this objective

1. 1 *is a strict unit*,
2. L^* *satisfies the left cancellation law*,
3. *the ordering* $\alpha \leqslant \beta$ *iff* $\alpha = \beta\gamma$ *for some* $\gamma \in L^*$ *is an inf-semilattice*.

Here L is obtained from L^* by adjoining to it an absorbent element 0. The last section shows that these conditions are sufficient for L-fuzzy sets of carrier X to be subsets of X in this new universe of usual sets.

This is a first attempt in a series of papers which will give concrete presentations of results and comments made in [6,10,11] where the "abstract nonsense" (as has been called category theory [14])