Available online at www.sciencedirect.com
SCIENCE DIRECT
Fuzzy Sets and Systems 149 (2005) 235–273
FUZZY sets and systems
www.elsevier.com/locate/fss

On fuzzy type theory☆

Vilém Novák*

Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic

Received 28 April 2003; received in revised form 5 November 2003; accepted 30 March 2004

Abstract

This paper is a generalization of classical (simple) type theory. We have developed a formal system of fuzzy type theory which differs from the classical one essentially in extension of truth values from two to infinitely many. The structure of truth values is assumed to be an IMTL-algebra (residuated lattice with prelinearity and double negation) extended by the Baaz delta operation. Various properties of fuzzy type theory are proved including its completeness.

Keywords: Fuzzy logic in narrow sense; Type theory; IMTL-algebra; IMTL logic; Higher order logic

1. Motivation

Fuzzy logic has been successfully developed including its predicate version especially thanks to Hájek and several other authors—see [9,12,24,25]. One of the main arguments in favour of it consists in its ability to provide a working model of some manifestations of the vagueness phenomenon. The latter is encountered in all kinds of human reasoning and is a distinguished feature of the semantics of natural language.

The fact that fuzzy logic touched the vagueness phenomenon led to a large number of its successful applications. Quite often, they rely on a simple model of the meaning of some words of natural language (cf., e.g. [23]). From linguistic point of view, however, these applications are more or less naive. Moreover, since fuzzy logic has been developed only up to first order, these applications are rather limited. If we want to apply fuzzy logic more deeply in linguistic semantics, higher order formulas have to be employed. This need comes out also in connection with the task to develop a semantic web.

☆ The paper has been supported by Grant 201/04/1033 of the GA ČR.
* Corresponding author. Tel.: +420-597-460-234; fax: +420-596-120-478.
E-mail address: vilem.novak@osu.cz (V. Novák).

0165-0114/$ - see front matter
doi:10.1016/j.fss.2004.03.027

In the accepted linguistic theory, an important role is played by intensional logic (cf. [1,9,?]), which is based on a simple type theory. Thus, a challenge is raised whether the latter could be generalized to fuzzy one.

Type theory has been originated by B. Russel and A. Whitehead in their *Principia matematica*. This theory has various versions, which can essentially be divided into two (overlapping) groups: a classical type theory being a classical logic of higher order, and a constructive type theory being a theory based on the corresponding constructive higher order logic and having extensive applications in theoretical computer science (e.g. analysis of correctness of programs, automated proving) or constructive mathematics. The first group has been initiated especially by Church [6] (due to him, classical type theory is often called Church's type theory) and Henkin [14,15] and continued in works of Andrews [2,3,4,16] and other authors (cf. citations therein). The second group covers a lot of publications which often refer to the seminal works of Martin-Löf [20]. Further on this direction—see [5,10,11] and the citations therein. There are also applications of the constructive type theory in linguistics (cf. e.g. [27]).

In this paper, we will formulate *fuzzy type theory* (FTT) as a *fuzzy logic of higher order* (and thus, belonging to the first of the above discussed groups). The set of truth values is assumed to form an IMTL algebra (on $[0,1]$, this is algebra of left continuous t-norms with involutive negation). We will follow the way of the development of the classical type theory as elaborated especially by A. Church and L. Henkin where the equivalence/equality belong among the basic connectives. In the fuzzy case this means that the equivalence is many-valued binary function and the equality is a fuzzy equality. Unfortunately, the latter leads to problems, which cannot be solved in full generality since equality, as the basic connective, must fulfil axioms which are valid for all formulas. In many-valued case, however, this is impossible and in most cases it finally forces the fuzzy equality to be classical crisp one. The only way which turned out possible was to introduce in the language also a special Δ connective which puts to 0 all truth values smaller than 1. Consequently, the fuzzy type theory presented in this paper is based on a linearly ordered IMTL_Δ algebra. This decision enabled us to preserve elegance of classical type theory. Generalizations to other kinds of structures of truth values is possible but it should be done carefully keeping in mind the overall purpose of such theory.

2. Syntax and semantics of fuzzy type theory

2.1. Introduction

Essential concept in fuzzy type theory is that of *type*. Informally, types can be understood as general characteristics of formulas which precisely determine their interpretation in the semantics; namely, they are used to distinguish special sets with specific properties. A formula of the given type is then interpreted as an element belonging to the set of the given type. In computer science, type often refers to a certain data structure, such as array, record, etc. We use Greek letters to denote types.

There are elementary and complex types. Formulas of elementary types represent objects from the given sets while complex types represent functions. Thus, if α and β are types then a formula A_α of type α represents some object from a set M_α (for precise definitions see below) and similarly,